Introduction to Character Theory

Seth Nelson

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Chapter 1

Introduction

This set of notes represents a rather unusual introduction to representation theory and character theory. This work is unusual in two senses. First, the introduction largely represents my own early approach using modules, when I was trying to understand what on earth representations were about and how one is supposed to use them. Nowadays, if I were to produce a new set of notes, I would do everything over k-algebras, the far more standard and modern approach.

Second, these notes were crafted while I was attempting to understand and reproduce the arguments from Udo Ott's work on generalized quadrangles [3]. As such, their purpose was to supply the theoretical background need to understand whatever Ott was saying. In practice, this fact means that I only cover the most basic and essential facts from character theory, and I only provide the representation theory necessary to understand the character theory. I do nothing more than this.

If I can say anything defensible about this work, it is that I do not assume too much from the reader. I tried to craft this text with a reasonably general audience in mind, and assume mostly that the reader is familiar with groups, rings, modules (though even this is unnecessary), and all the essential stuff from algebra. I was an advanced undergrad when I wrote this note set, and so perhaps it will be useful to people who are just now starting to work on character theory.

What I do cover consists of a basic introduction to modules, Artinian modules, finitely generated modules, irreducible and completely reducible modules, and Schur's lemma. I also cover Maschke's Theorem, as is necessary. My section on representation theory is essentially an extension of what I do with modules. For characters, I prove in full the first and second orthogonality relations, and all of the basic facts about irreducible characters. I give special treatment to linear characters at the end.

Finally, some of what I do is my own work, and some of what I do is borrowed from textbooks. I often follow Martin Isaac's work *Character Theory of Finite Groups* [1], James and Liebeck's book *Representations and Characters of Groups* [2], and Serre's book *Linear representations of finite groups* [4]. Of course, I provide a citation to the relevant theorems every time.

Chapter 2

Module Theory

Modules are often overlooked in a standard introductory algebra course, but they are fundamental to the study of algebra, and they will prove to be especially important in our discussion of character theory. Generally, representation theory is developed over k-algebras, a special class of module, but we begin in a slightly different framework.

2.1 Modules, Module Homomorphisms, and Submodules

In general, a module is an additive group which is equipped with a scalar action by a ring. Throughout this thesis, we will always assume that a ring contains an identity element, denoted by 1.

Definition 2.1. A *left R-module* $(M; R, +, \cdot)$ consists of an abelian group $(M, +)$ together with a ring R which acts on M from the left. To be precise, if $r \in R$ and $a \in M$ then $r \cdot a$ is also in M. Formally, we think of this action as a map $\cdot : R \times M \longrightarrow M$ which satisfies the following properties, for all $r, s \in R$ and all $a, b \in M$.

(a) Distributivity of multiplication over module addition, $r \cdot (a + b) = r \cdot a + r \cdot b$.

(b) Distributivity of multiplication over scalar addition, $(r + s) \cdot a = r \cdot a + s \cdot a$.

(c) Associativity of scalar multiplication, $(rs) \cdot a = r \cdot (s \cdot a)$.

(d) A scalar identity, $1 \cdot a = a$.

Generally, we will drop the notation $\cdot$, and write ra for $r \cdot a$

Note that one may also define right R-modules, in which our ring R acts on M from the right. That is, $r \in R$ acts on $a \in M$ by $a \cdot r$ instead of $r \cdot a$. We can likewise define double sided modules where R acts on M from both the right *and* the left. In this text, modules are assumed to be left modules. We should mention as well that we often speak about a module M "over a ring R." This is common language, and it simply means that R acts on M. Additionally, R is occasionally referred to as a *ring of scalars*, since R is a ring consisting of scalars which act on M. If the concept of a scalar reminds our reader of a vector space, then they are headed in the right direction.

Example 2.2. The most fundamental example – in fact the motivating example – of a module is a vector space over a field F.

Example 2.3. Consider the ring of integers $\mathbb{Z}$, and the ideal $2\mathbb{Z} \subseteq \mathbb{Z}$. Now, $2\mathbb{Z}$ is closed under addition, since if $n, m \in 2\mathbb{Z}$, then $n = 2\ell$, $m = 2k$, and so $n + m = 2\ell + 2k = 2(\ell + k) \in 2\mathbb{Z}$. Of course, every element in $2\mathbb{Z}$ also has an additive inverse in $2\mathbb{Z}$, and we have $0 \in 2\mathbb{Z}$. Finally, if $r \in \mathbb{Z}$, then for all $a \in 2\mathbb{Z}$, we have $r \cdot a \in 2\mathbb{Z}$, since $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$. The fact that $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$ also proves axioms (a) – (d) of a module. Thus, $2\mathbb{Z}$ is a $\mathbb{Z}$-module. One may extend this argument to any ideal $n\mathbb{Z} \subseteq \mathbb{Z}$, or, indeed, to any ideal inside any ring.

The next example is important enough to get its own definition. This example is foundational to the study of representation theory, but it is perhaps the simplest example we provide.

Definition 2.4. Let R be any ring. Then, R is an R-module over itself. This module is known as the *regular R-module*, and we denote it by R°.

Now, R° is technically distinct from the ring R. In R°, only addition is defined. All multiplication operations happen by the action of R on R°. Indeed, if $s \in R^\circ$, and $r \in R$. Then, $rs \in R^\circ$, and the four module axioms (a)–(d) also follow immediately from the axioms for a ring.

Definition 2.5. Suppose that M and N are left R-modules. Then, an *R-homomorphism* or *R-module homomorphism* $\phi : M \longrightarrow N$ is a function such that for all $a, b \in M$ and $r \in R$, we have

$$\phi(a + b) = \phi(a) + \phi(b)$$

and

$$\phi(r \cdot a) = r \cdot \phi(a).$$

When the ring R is understood, we drop the designation R-homomorphism and simply write *homomorphism.*

Definition 2.6. For two R-modules, M and N, we define $\mathrm{Hom}_R(M, N)$ to be the set of R-module homomorphisms $\phi : M \longrightarrow N$.

Now, $\mathrm{Hom}_R(M, N)$ is again a module. Indeed, we may define R to act on $\mathrm{Hom}_R(M, N)$ by defining $r \cdot \phi$ to be $r \cdot \phi(a) = r\phi(a)$, and we define addition by assigning $\phi + \psi$ to the function $(\phi+\psi)(a) = \phi(a)+\psi(a)$. These are all homomorphisms, turning $\mathrm{Hom}_R(M, N)$ into an R-module.

Definition 2.7. Let M be an R-module, and suppose that N is a subset of M which is closed under addition and scalar multiplication. Then, N is a *submodule* of M.

Example 2.8. Recall once again the $\mathbb{Z}$-module $2\mathbb{Z}$. Now, note that $4\mathbb{Z} \subseteq 2\mathbb{Z}$. Of course, $4\mathbb{Z}$ is also a $\mathbb{Z}$-module, and so $4\mathbb{Z}$ is a submodule of $2\mathbb{Z}$.

One may further show that if M and N are left R-modules, and there exists an R-homomorphism $\phi : M \longrightarrow N$, then for any submodule $W \subseteq M$, the image $\phi(W)$ is a submodule of N.

2.2 Irreducible Modules, Reducible Modules, Direct Sums

We now turn to the notion of reducibility and irreducibility. It turns out that most of the important modules we will want to study are especially nice modules, which may be described entirely by a collection of irreducible submodules. This section is dedicated to making that idea precise.

Definition 2.9. Let M be an R-module. Then M is called *irreducible* if $M \neq \{0\}$ and every submodule of M is either M or $\{0\}$.

Example 2.10. For example, the modules $\mathbb{Z}/p\mathbb{Z}$ are all irreducible as $\mathbb{Z}$-modules. On the other hand, $\mathbb{Z}/n\mathbb{Z}$ are not irreducible as $\mathbb{Z}$-modules if n is composite.

The following is an important fact about irreducible modules, which we will use often.

Lemma 2.11. *Let M be an R-module. Let I be an ideal of R. Then, for $a \in M$, $Ia = \{r \cdot a : r \in I\}$ is an R-module. If M is irreducible, $I = R$, and $a \neq 0$, then $Ra = M$.*

Proof. First, Ia is a submodule. For if $b, c \in Ia$, then $b = ra$ and $c = sa$, with $r, s \in I$, so $(r + s) \in I$, and thus $b + c = (r + s)a \in Ia$. Moreover, if $b = ra \in Ia$, then $r \in I$, so for all $s \in R$, we $s \cdot b = (sr)a \in Ia$ because $sr \in I$. Thus, Ia is a submodule as claimed. Now, if $a \neq 0$, $I = R$, and M is irreducible, then $1 \cdot a = a \in Ra$, so Ra is a nonzero submodule of M, and thus $Ra = M$, since M is irreducible. □

We now prove the famous Schur's Lemma, which is a consequence of Lemma 2.11.

Lemma 2.12 (Schur's Lemma)**.** *Let M, N be irreducible left R-modules. Suppose there is a map $\phi : M \longrightarrow N$, such that $\phi(M) \neq \{0\}$. Then, $M \cong N$.*

Proof. Take $a \in M$ such that $\phi(a) \neq 0$. Then, $\phi(a) \in N$, so $\phi(Ra) = R\phi(a) = N$ by Lemma 2.11, given that $\phi(a) \neq 0$. Moreover, since $\phi(M) \neq \{0\}$, then $\ker \phi \neq M$, forcing $\ker \phi = \{0\}$, by the irreducibility of M. Thus, $M \cong M/\ker \phi \cong N$, as claimed. □

Suppose that M, N are R-modules. Then, we may produce a new R-module, $\{(m, n) : m \in M, n \in N\}$ consisting of ordered pairs from M and N, where $r \cdot (m, n) := (rm, rn)$ and $(m_1, n_1) + (m_2, n_2) := (m_1 + m_2, n_1 + n_2)$. The module is denoted as $M \oplus N$, and we refer to it as the *direct sum* of M and N.

Definition 2.13. We call a module V *completely reducible* if for every submodule $M \subseteq V$, there exists a submodule N of V such that $M \oplus N \cong V$.

In this case, the module N is called *complementary to M* or the *complement of M.*

Example 2.14. All modules V over a field F are completely reducible. Indeed, in this case, V is an F-vector space, so for any $M \subseteq V$, M has a basis $\{v_1, \ldots, v_k\}$ which may be completed to a basis $v_1 \ldots, v_k, \ldots, v_n$ for all of V. Then, $\text{Span}\{v_{k+1}, \ldots, v_n\}$ is complementary to N.

Completely reducible modules are extremely nice modules. As we will see, if a module V is completely reducible, and also satisfies some special finiteness conditions, then the structure of V is entirely determined by the set of irreducible modules N_i in V. The next section is entirely dedicated to formalizing this idea. For now, though, we will prove some facts about direct sums, which will prove useful to us now and later on.

Lemma 2.15. *If V is an R-module, then $V \cong S \oplus T$ if and only if there exist $M, N \subseteq V$ such that $M \cap N = \{0\}$ and $M + N = V$ satisfying $M \cong S$ and $N \cong T$.*

Proof. First, suppose that $M, N \subseteq V$ exist satisfying $M \cap N = \{0\}$ and $M + N = V$. Define $\phi : M \oplus N \longrightarrow V$ by $(m, n) \mapsto m + n$. This is an R-module homomorphism, for if $r \in R$, then $\phi(r \cdot (m, n)) = r \cdot \phi(m + n)$. If $m_1, m_2 \in M$, $n_1, n_2 \in N$, then $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$, and

$$\phi((m_1 + m_2, n_1 + n_2)) = (m_1 + n_1) + (m_2 + n_2) = \phi((m_1, n_1)) + \phi((m_2, n_2)),$$

so $\phi((m_1, n_1) + (m_2, n_2)) = \phi((m_1, n_1)) + \phi((m_2, n_2))$. Moreover, ϕ is injective, for if $\phi((m, n)) = 0$, then $m + n = 0$, so that $n = -m \in M$. Therefore, $n \in N \cap M = \{0\}$, so $n = 0$, forcing $m = 0$, and thus $\ker \phi = \{(0, 0)\}$. Finally, ϕ is surjective. For if $v \in V$, then by assumption $v = m + n$ for some $m \in M$ and $n \in N$. Therefore, $\phi((m, n)) = v$. Then, we have constructed an isomorphism $S \oplus T \cong V$, where $M \cong S$ and $N \cong T$.

Now, suppose that $S \oplus T \cong V$. Then, we have a left R-module isomorphism $\phi : S \oplus T \longrightarrow V$. Moreover, there are two injections $\iota_S : S \longrightarrow S \oplus T$ and $\iota_T : T \longrightarrow S \oplus T$ by $\iota_S(s) = (s, 0)$ and $\iota_T(t) = (0, t)$. Since ϕ is an isomorphism, then $M = \phi \circ \iota_S(S)$ and $N = \phi \circ \iota_T(T)$ are both submodules of V. Moreover, $M \cong S$ and $N \cong T$, for $\phi \circ \iota_S$ and $\phi \circ \iota_T$ are injections, and these of course surject onto $M = \phi \circ \iota_S(S)$ and $N = \phi \circ \iota_T(T)$. We now prove $M \cap N = \{0\}$ and $M + N = V$. First, $M \cap N = \{0\}$, for if $a \in M \cap N$, then $\phi^{-1}(a) \in \iota_S(S) \cap \iota_T(T)$, where $\iota_S(S) \cap \iota_T(T) = \{(0, 0)\}$, so that $\phi^{-1}(a) = (0, 0)$, and thus $a = 0$ as needed. Finally, $M + N = V$. Taking $v \in V$, then $\phi^{-1}(v) = (s, t)$ for some s and t. Therefore, $v = \phi((s, t)) = \phi((s, 0)) + \phi((0, t))$, with $\phi((s, 0)) = \phi \circ \iota_S(s) \in M$ and $\phi((0, t)) = \phi \circ \iota_T(t) \in N$, proving $v \in M + N$, as needed. □

Remark 2.16. Now, say that V is an R-module such that $V = N_1 \oplus N_2 \oplus \cdots \oplus N_r = \bigoplus_{i=1}^r N_i$. Then, using induction, we may show that there exist submodules $M_1, \ldots, M_r$ of V with $M_i \cong N_i$ such that $V = M_1 + M_2 + \cdots + M_r = \sum_{i=1}^r M_i$.

We now prove Lemma 2.17, which will give us a method to construct direct sums out of modules.

Lemma 2.17. *Let V be a completely reducible R-module. Suppose that there exists a surjective map $\psi : V \longrightarrow M$. Then, there exists some $S, T \subseteq V$ with $S \cong M$ such that $V \cong S \oplus T$.*

Proof. We show that $V \cong M \oplus \ker \psi$. Now, $\ker \psi$ is a submodule of V, and since V is completely reducible, then there exists a complementary submodule N such that $V \cong N \oplus \ker \psi$. By Lemma 2.15, V contains submodules S, T such that $S \cong N$ and $T \cong \ker \psi$ with $S + T = V$ and $S \cap T = \{0\}$. We claim that $S \cong M$. Indeed, there is a projection $\pi : V \longrightarrow V/T$, and since $T \cong \ker \psi$, then $V/T \cong V/\ker \psi$. By the First Isomorphism

Theorem, $V/\ker\psi \cong M$. We show that $S \cong V/T$ to prove $S \cong M$. So, take $\phi : S \longrightarrow V/T$ by $s \mapsto s + T$. This is a homomorphism, so we show it is a bijection. Since $S \cap T = \{0\}$, then ϕ is an injection, for if $\phi(s) = 0 + T$, then $s = s - 0 \in T$, forcing $s = 0$. On the other hand, taking $v + T \in V/T$, we have $v = s + t \in V$, so $v + T = s + t + T = s + T$, and thus $\phi(s) = v + T$, proving ϕ is surjective. Thus, ϕ is an isomorphism, so $S \cong V/T \cong M$. Therefore, $V \cong S \oplus T \cong M \oplus \ker\psi$ as claimed. □

Using Lemma 2.17, we have the following very nice fact, for which we have used Isaacs [1, Lemma 1.14].

Proposition 2.18. *Let R be a ring, and suppose that R° is completely reducible. Then, every irreducible R-module V is isomorphic to a submodule of R°.*

Proof. Let V be an irreducible R-module. Pick $v \in V$ with $v \neq 0$, and define $\phi : R^\circ \longrightarrow V$ by $r \mapsto rv$. Then ϕ is an R-module homomorphism, for if $r, s \in R^\circ$, then $\phi(r+s) = (r+s)v = rv + sv = \phi(r) + \phi(s)$. Moreover, $\phi(r \cdot s) = (r \cdot s)v = r(sv) = r\phi(s)$. Since V is irreducible, then by Schur's Lemma, $\phi(R^\circ) = V$, proving ϕ is an $R^\circ \longrightarrow V$ surjection. Thus, by Lemma 2.17, $R^\circ \cong V \oplus \ker\phi$. Therefore, by Lemma 2.15, V is isomorphic to a submodule of R°. □

This proposition is hugely important, and we will use it quite often. It tells us that when the regular module of a ring R is completely reducible, then it contains all irreducible submodules R. Hopefully, the reader has begun to see the importance of the regular module of a ring R, as well as the usefulness of complete reducibility.

2.3 Modules over Artinian Rings

Artinian rings are rings with a finiteness condition placed on their ideals. They are important, for example, when one would like to find such things as a minimal nonzero ideal of a given ring. In fact, we will use this property to show that every module over an Artinian ring contains an irreducible submodule.

Definition 2.19. Let R be a ring. Then R is called a *left Artinian ring* if every infinite descending chain $I_1 \supseteq I_2 \supseteq I_3 \ldots$ of left ideals stabilizes. That is, there exists some k such that for all $m \geqslant k$, $I_k = I_m$.

Example 2.20. We enumerate some examples of Artinian and non-Artinian rings.

(a) All fields F are left Artinian, for their ideals consist of $\{0\}$ and F.

(b) Any finite ring is left Artinian.

(c) $\mathbb{Z}$ is not left Artinian. For every ideal $n\mathbb{Z} \subseteq \mathbb{Z}$, there exists some $m = kn$ for $k > 1$ such that $m\mathbb{Z} \subseteq n\mathbb{Z}$.

Just as with modules, we may consider left Artinian, right Artinian, or double-sided Artinian rings. When the ring in question is commutative, there is no distinction between any of these.

Lemma 2.21. *Suppose that V is a nontrivial R-module for a left Artinian ring R. Then, V contains at least one irreducible submodule.*

Proof. Since R is left Artinian, every list of distinct ideals $I_1, I_2, I_3, \ldots$ satisfying $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ between R and $\{0\}$ is finite. So, set $I_1 = R$ and let $I_1, \ldots, I_n$ be a maximal such list between R and $\{0\}$. Now, take $v \in V$ with $v \neq 0$, and consider the list $I_1 v, \ldots, I_n v$, which are all R-modules by Lemma 2.11. Pick I_m so that m is the largest index for which $I_m v$ is nonzero. Some index m exists, for taking $m = 1$, we have $I_1 v = Rv$ which contains the nonzero element $1 \cdot v = v$. Let N be a submodule of $I_m v$. Then, each element of N takes the form av for some $a \in I_m$, so define $J = \{a \in I_m : av \in N\}$.

We claim that J is a left ideal. So take $a \in J$. Then, $av \in N$, so for all $r \in R$, we have $r(av) = (ra)v \in N$, given that N is a left R-module. Thus, $ra \in J$ for all $r \in R$. Moreover, if $a, b \in J$, then $av, bv \in N$, so $av + bv = (a + b)v \in N$, and thus $a + b \in J$. Therefore, J is a left ideal as we claimed. Moreover, $J \subseteq I_m$ by definition. By the maximality of the list $I_1, \ldots, I_n$, we must have $J = I_k$ for $m \leqslant k \leqslant n$. Now, if $k = m$, then $N = Jv = I_m v$. Otherwise, if $k > m$, then $N = Jv = I_k v = \{0\}$. Thus, for N an arbitrary submodule of $I_m v$, we have either $N = I_m v$ or $N = \{0\}$. We conclude then that $I_m v$ is irreducible. □

We have now developed the sufficient machinery to prove our rather ambiguous statement in Section 2.2 that certain classes of completely reducible modules are totally determined by their irreducible submodules. In particular, we show that if a module V is completely reducible over an Artinian ring R, then V is a direct sum of a collection of irreducible submodules of V. Compare the following lemma to [1, Theorem 1.10].

Lemma 2.22. *Let V be an R-module for a left Artinian ring R. Then, if V is completely reducible, there exist a set of irreducible modules $\{N_\alpha\}_{\alpha \in J}$ in V such that $V \cong \bigoplus_{\alpha \in J} N_\alpha$.*

Proof. Since V is a module over an Artinian ring R, there is at least one irreducible submodule N of V by Lemma 2.21. So, take a composition $\bigoplus_{\alpha \in J} N_\alpha$ of submodules which is maximal in the sense that there is no disjoint irreducible submodule W such that $\bigoplus_{\alpha \in J} N_\alpha \oplus W$ is a new larger composition. Since V is completely reducible, there exists some M such that $V \cong (\bigoplus_{\alpha \in J} N_\alpha) \oplus M$. The modules M and $\bigoplus_{\alpha \in J} N_\alpha$ are disjoint, since M is the kernel of the projection $V \longrightarrow \bigoplus_{\alpha \in J} N_\alpha$. Since M is itself an R-module, then M contains some irreducible submodule W by Lemma 2.21 which is not one of the N_α, so that $\bigoplus_{\alpha \in J} N_\alpha \oplus W$ is a new composition of irreducible submodules. This contradicts the maximality of $\bigoplus_{\alpha \in J} N_\alpha$, so $V \cong \bigoplus_{\alpha \in J} N_\alpha$. □

2.4 The Group Algebra

In our study of character theory, we will primarily be occupied with the group algebra, which forms an important module closely intertwined with the structure of its defining group. Now, in principle the group algebra can be defined for any ring R. We need this definition, for we use it later in our thesis. But, the reader should bear in mind that our most important case is when we consider a group algebra over $\mathbb{C}$ or $\mathbb{Q}(\zeta)$ for ζ an nth root of unity.

Definition 2.23. Let R be ring, and let G be a group. Label the elements of G by $g_1, \ldots, g_n$, and formally define the set of commutative products $ag_i = g_i a$ for all $a \in R$. Then, define the *group algebra* $R[G]$ to be the set of all "formal sums" $\sum_{i=1}^n a_i g_i$ for $a_i \in R$. For two elements $\sum_{i=1}^n a_i g_i$ and $\sum_{i=1}^n b_i g_i$ of $R[G]$, define their sum by

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i.$$

Note that we have defined this addition operation such that $R[G]$ is an abelian group.

We may also define multiplication in the group algebra by

$$\sum_{i=1}^{n} a_i g_i \cdot \sum_{j=1}^{n} b_j g_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j) g_i g_j,$$

where $g_i g_j$ is computed in G, and $a_i b_j$ is computed in R. Now, this multiplication is distributive and associative as expected – although, depending on R and G, it may not be commutative.

To be clear, when we say "formal sums", we mean that the sum $g_i + g_j \in R[G]$ does not imply any new addition operation sending $g_i + g_j$ to some new group element in G. In fact, in the case when $R = F$ is a field, we tend to think of the elements $g_1, \ldots, g_n \in F[G]$ as basis elements for $F[G]$. That is, $a_1 g_1 + \cdots + a_n g_n$ generates all elements of $F[G]$ by F-linear sums. Thus, $F[G]$ is an F-vector space and an F-module. It is, moreover, a ring, since both addition and multiplication are defined on it.

Remark 2.24. Generalizing our discussion of basis vectors, if we consider $R[G]$ as an R-module, or as the regular module over itself, then $R[G]$ is finitely generated whenever G is a finite group.

We now specialize to group rings $R[G]$ for which $R = F$ is a field. These are especially nice group rings, and, luckily, most of representation theory consists of analyzing these group rings, particularly in the case $F = \mathbb{C}$.

Lemma 2.25. *For any field F and any finite group G, $F[G]$ is a left Artinian ring.*

Proof. Consider a descending chain of left ideals $I_1 \supseteq I_2 \supseteq I_3 \ldots$. Each I_n is an F-vector space over a set of G basis vectors. Therefore, $\dim I_n$ decreases monotonically with n, and $\dim I_n$ is bounded below by 0. So there is some N such that the sequence $\dim I_1, \dim I_2, \dim I_3, \ldots$ stabilizes at $k = \dim I_m$ for all $m \geqslant N$. Thus, for all $n, m \geqslant N$, with $I_n \supseteq I_m$, we have $\dim I_n = \dim I_m$ and thus $I_n = I_m$. So, $I_1 \supseteq I_2 \supseteq I_3 \ldots$ stabilizes, proving $F[G]$ is left Artinian. $\square$

Now, suppose additionally that F is an algebraically closed field. Then, we have the following rather strong modification of Schur's Lemma. The reader should note that this next result is also sometimes called Schur's Lemma. Here, we follow James & Liebeck [2, 9.1].

Lemma 2.26. *Let M be an irreducible $F[G]$-module, for F an algebraically closed field. Then, for every isomorphism $\phi : M \longrightarrow M$, there exists some $\lambda \in F$ such that $\phi = \lambda \mathrm{id}_M$.*

Proof. Regarding M as an F-vector space, ϕ is an F-linear endomorphism of M. Since F is algebraically closed, ϕ has an eigenvalue, $\lambda \in F$. Therefore, $\ker(\phi - \lambda \mathrm{id}_M) \neq \{0\}$. Thus, by Schur's Lemma, $\ker(\phi - \lambda \mathrm{id}_M) = M$, so $\phi = \lambda \mathrm{id}_M$. □

2.5 Decomposing the Group Algebra and Maschke's Theorem

In this section, we prove the all-important Maschke's Theorem, which shows that modules over rings $F[G]$ are a direct sum of some of their irreducible submodules. We also prove a collection of important conditions describing exactly what this decomposition looks like. For Maschke's Theorem, we closely follow Isaacs [1, Theorem 1.9].

Theorem 2.27 (Maschke's Theorem)**.** *Let G be a finite group and F a field whose characteristic does not divide $|G|$. Then every $F[G]$-module is completely reducible.*

Proof. Let V be an $F[G]$-module with the $F[G]$-submodule $M \subseteq V$. Now, V is also an F-module and thus an F-vector space, so that M is an F-subspace of V. Since vector spaces are completely reducible, then there exists an F-submodule N_0 which is complementary to M. Therefore, we may define a projection $\pi : V \longrightarrow M$, where $\ker \pi = N_0$. We will modify π to create an $F[G]$-homomorphism $\psi : V \longrightarrow M$, which projects from V to M. By Lemma 2.17, M then has a complement in V, so since M was an arbitrary submodule of V, we have that V is completely reducible.

We now define our new projection,

$$\psi(v) := \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gv).$$

It is challenging to motivate why we have chosen ψ for our new map. The best motivation we are able to provide is that this sort of "averaging trick" (as it is called) is common in group theory whenever one hopes to build some map ψ which is somehow invariant under G.

Now, we need to verify that ψ is a projection and an $F[G]$-homomorphism. Indeed, for $a \in F$, given that a commutes with $g \in G$ and π is an F-homomorphism,

$$\psi(av) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(g(av)) = a \left(\frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gv) \right) = a\psi(v).$$

In the same way, if $v, w \in V$, then

$$\psi(v + w) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(g(v + w)) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gv) + \frac{1}{|G|} \sum_{g \in G} g^{-1} \pi(gw) = \psi(v) + \psi(w),$$

since, once again, π is a module homomorphism.

We now prove that ψ is linear with respect to $h \in G$.

$$\psi(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\pi(g(hv)),$$

and since g runs over all elements of G, we substitute g for kh^{-1} and run along k in G,

$$\begin{aligned}
\psi(hv) &= \frac{1}{|G|} \sum_{k \in G} (kh^{-1})^{-1}\pi(kh^{-1}hv) \\
\psi(hv) &= \frac{1}{|G|} \sum_{k \in G} hk^{-1}\pi(kv) \\
\psi(hv) &= h\left(\frac{1}{|G|} \sum_{k \in G} k^{-1}\pi(kv)\right) \\
\psi(hv) &= h\psi(v)
\end{aligned}$$

so ψ is an $F[G]$-homomorphism.

We now show ψ is a projection onto M. Let $m \in M$, then for all $g \in G$, $gm \in M$, so $\pi(gm) = gm$. Thus,

$$\psi(m) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\pi(gm) = \frac{1}{|G|} \sum_{g \in G} g^{-1}gm = \frac{1}{|G|} \sum_{g \in G} m = m$$

So, ψ is a projection onto M. Therefore, by Lemma 2.17, M has a complement N in V. Since M was an arbitrary $F[G]$-submodule of V, we conclude that V is completely reducible. □

Proposition 2.28. *Let V be an $F[G]$-module. Then, there is some set $\{N_\alpha\}_{\alpha \in J}$ of irreducible submodules indexed by J such that $V \cong \bigoplus_{\alpha \in J} N_\alpha$.*

Proof. The statement follows from Lemma 2.22 and Lemma 2.25. □

In the case that our $F[G]$-module V equals $F[G]$, we would like to prove some facts about what the decomposition $V \cong \bigoplus_{\alpha \in J} N_\alpha$ looks like. Namely, we want to show that the number $|J|$ of irreducible modules in the factorization is finite, and that the number of times a given module N appears in the decomposition equals $\dim_F N$.

Proposition 2.29. *Let F be a field, G a finite group, and consider a finitely generated $F[G]$-module V, where $V \cong \bigoplus_{\alpha \in J} N_\alpha$. Then, the indexing set J is finite.*

Proof. Since V is finitely generated, then V is a finite dimensional F-vector space, which decomposes into the direct sum of vector spaces $\bigoplus_{\alpha \in J} N_\alpha$. Thus, $\dim_F V$ is bounded below by $|J|$. Since $\dim_F V$ is finite, then J is finite. □

For the next proposition, we will need a few new facts about the set of module homomorphisms between two R-modules M and N.

Remark 2.30. Recall that for two R-modules, M and N, we define $\mathrm{Hom}_R(M, N)$ to be the set of R-module homomorphisms $\phi : M \longrightarrow N$. Note that $\mathrm{Hom}_R(M, N)$ is an R-module itself. Indeed, if $\phi, \psi : M \longrightarrow N$, then $(\phi + \psi)(a) := \phi(a) + \psi(a)$ and $r\phi(a) := \phi(ra)$ give module operations on $\mathrm{Hom}_R(M, N)$. Moreover, for any set of R-modules $N_1, N_2, \ldots, N_r$, and any R-module M, we have $\mathrm{Hom}_R(\bigoplus_{i=1}^r N_i, M) \cong \bigoplus_{i=1}^r \mathrm{Hom}_R(N_i, M)$. This is an R-module isomorphism under the mapping $\psi \mapsto (\psi|_{N_1}, \psi|_{N_2}, \ldots, \psi|_{N_r})$. We will use this isomorphism freely from now on.

For this next proof, we once again follow James & Liebeck [2, Proposition 11.8].

Proposition 2.31. *Let F be an algebraically closed field, G a finite group, and consider the decomposition $F[G] \cong \bigoplus_{i=1}^m N_i$ for an irreducible set of R-modules $N_1, \ldots, N_m$. Fix a particular irreducible module $M \subseteq F[G]$. Then, the number of copies $M_1, \ldots, M_n$ of M in the decomposition $\bigoplus_{i=1}^m N_i$ equals $\dim_F M$.*

Proof. First, we claim that $\dim_F \mathrm{Hom}_{F[G]}(F[G], M)$ equals the number of isomorphic copies $M_1, \ldots, M_n$ of M in the decomposition $\bigoplus_{i=1}^m N_i$. So,

$$\mathrm{Hom}_{F[G]}(F[G], M) \cong \mathrm{Hom}_{F[G]}(\bigoplus_{i=1}^m N_i, M) \cong \bigoplus_{i=1}^m \mathrm{Hom}_{F[G]}(N_i, M).$$

By Schur's Lemma, for every $N_i \ncong M$, $\mathrm{Hom}_{F[G]}(N_i, M)$ consists only of the 0 map. Thus, for the full set $M_1, \ldots, M_n \cong M$ in $N_1, \ldots, N_m$, we have

$$\bigoplus_{i=1}^m \mathrm{Hom}_{F[G]}(N_i, M) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{F[G]}(M_i, M).$$

By Lemma 2.26, each of $\mathrm{Hom}_{F[G]}(M_i, M)$ are 1 dimensional over F since F is algebraically closed. Therefore, $\dim_F \bigoplus_{i=1}^n \mathrm{Hom}_{F[G]}(M_i, M) = n$, the number of copies $M_1, \ldots, M_n$ isomorphic to M.

Now, we show that $\dim_F M = \dim_F \mathrm{Hom}_{F[G]}(F[G], M)$. Let $v_1, \ldots, v_d$ be a $\mathbb{C}$-basis for M. Define $\phi_i : F[G] \longrightarrow M$ by $\phi(a) = av_i$ for $a \in F[G]$. This is an $F[G]$-homomorphism, for if $a, b \in F[G]$, then $\phi_i(a + b) = (a + b)v_i = av_i + bv_i = \phi_i(a) + \phi_i(b)$. Moreover, for $r \in F[G]$, we have $\phi_i(ra) = rav_i = r(av_i) = r\phi_i(a)$. We now show that the ϕ_i form a $\mathbb{C}$-basis for $\mathrm{Hom}_{F[G]}(F[G], M)$. Take $\phi \in \mathrm{Hom}_{F[G]}(F[G], M)$, and $1 \in F[G]$. Then, $\phi(1) = \sum_{i=1}^d \lambda_i v_i$. Thus,

$$\phi(r) = r\phi(1) = r \sum_{i=1}^d \lambda_i v_i = \sum_{i=1}^d \lambda_i (rv_i) = \sum_{i=1}^d \lambda_i \phi_i(r),$$

so the $\phi_1, \ldots, \phi_d$ span $\mathrm{Hom}_{F[G]}(F[G], M)$. Moreover, if $\sum_{i=1}^d \lambda_i \phi_i = 0$, then $\sum_{i=1}^d \lambda_i \phi_i(1) = \sum_{i=1}^d \lambda v_i = 0$. Since the v_i form a basis, then $\lambda_1 = \ldots = \lambda_d = 0$, so the ϕ_i form a basis as well. Thus, $\dim_F M = \dim_F \mathrm{Hom}_{F[G]}(F[G], M)$, so the claim is proven. $\square$

The two results we have just proved are rather powerful. They tell us a lot about the decomposition of $F[G]^\circ$. If F is algebraically closed, then Proposition 2.31 tells us that every irreducible submodule of $F[G]$ appears in the decomposition, and it states exactly how many times this module appears. In conjunction with Proposition 2.29, this means that there are only a finite number of distinct irreducible modules in $F[G]$. Moreover, by Proposition 2.18, all possible irreducible $F[G]$-modules are isomorphic to a submodule of $F[G]^\circ$, so we have placed a heavy restriction on the irreducible $F[G]$-modules.

Remark 2.32. We mention one further fact about $F[G]^\circ$. Once again, for some irreducible N_i, we have $F[G]^\circ \cong \bigoplus_{i=1}^m N_i$. By Lemma 2.15, we may view each N_i as a submodule of $F[G]$, and by Remark 2.16, we then have $\sum_{i=1}^m N_i = F[G]$. We are then justified in viewing $\sum_{i=1}^m N_i$ and $\bigoplus_{i=1}^m N_i$ as interchangeable representations of the same module. Thus, each element a of $\sum_{i=1}^m N_i$ is a sum $\sum_{i=1}^n a_i$ in $F[G]$ with $a_i \in N_i \subseteq F[G]$. Recall that $F[G]$ is also a ring, and thus $F[G]$ contains an identity element 1. Then, by the preceding discussion, we have $1 = \sum_{i=1}^m e_i$ for $e_i \in N_i$. We refer to e_i as the N_i *component of* 1. Note that if we have an isomorphism $\phi : F[G]^\circ \longrightarrow \bigoplus_{i=1}^m N_i$, then $\phi(1) = (e_1, \ldots, e_n)$. It is in this sense that we refer to e_i as a component of 1.

Finally, $F[G]$ is a ring, so for any two elements a, b, the multiplication ab is defined. We claim that if $a \in N_i$, then $ae_i = a$ and $ae_j = 0$. So,

$$a = a \cdot 1 = a\left(\sum_{j=1}^n e_j\right) = \sum_{j=1}^n ae_j,$$

and

$$a = 1 \cdot a = \left(\sum_{j=1}^n e_j\right) a = \sum_{j=1}^n e_j a,$$

so that $\left(\sum_{j=1}^n e_j\right) a = \sum_{j=1}^n ae_j$. Thus, $\sum_{j=1}^n ae_j = \left(\sum_{j=1}^n e_j\right) a = a \in N_i$. On the other hand, each product ae_j is in N_j, forcing $ae_j = 0$. So, $a \cdot 1 = \sum_{j=1}^n ae_j = ae_i$. Note in particular that $e_j e_i = 0$ whenever $j \neq i$, otherwise $e_i^2 = e_i$.

Chapter 3

Representations and Characters

The representation theory of finite groups studies the action of groups on vector spaces. Over the past few sections, we have done everything over a field F, where F is sometimes algebraically closed. We now specialize to the case $F = \mathbb{C}$. In general, classifying the possible representations of a particular group is a hard problem. Character Theory provides a method of simplifying this problem greatly. For this next section, we follow Isaacs [1] especially closely, though we also use James & Liebeck [2] and Serre [4].

3.1 Representations and Modules

In this section, we introduce the notion of a representation, and then explore the connection between a representation of G and a module of the group algebra $\mathbb{C}[G]$. Hopefully, by the end of this section, you will begin to see why we spent so long discussing modules.

Definition 3.1. The multiplicative group $\mathrm{GL}_n(\mathbb{C})$ consists of all $n \times n$ invertible matrices over $\mathbb{C}$. If G is a group, then a $\mathbb{C}$-representation of G is a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

We may generalize the definition of a representation to any field. Indeed, an F-representation of G is a homomorphism $G \longrightarrow \mathrm{GL}_n(F)$. If $\mathfrak{F}$ is a representation of G in $\mathrm{GL}_n(\mathbb{C})$, then we say that $\mathfrak{F}$ has *degree n*. Note that $\mathrm{GL}_n(\mathbb{C})$ is just the multiplicative group of invertible $\mathbb{C}$-matrices, considered outside the context of a particular vector space. If we instead write $\mathrm{GL}(V)$ for an n-dimensional vector space V, then this is the group of invertible linear transformations of a vector space V, in which case it is necessary to specify a basis with respect to which the matrix transformations occur.

Remark 3.2. We have the following situation.

(a) If $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is a representation of G, then for all $g, h \in G$, $\mathfrak{F}(g), \mathfrak{F}(h)$ are invertible matrices satisfying $\mathfrak{F}(gh) = \mathfrak{F}(g)\mathfrak{F}(h)$. As promised, the homomorphism $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ corresponds naturally to some action of G on a vector space V. Indeed, set $V = \mathbb{C}^n$. Then for all $g \in G$ and $v \in V$, we define our action of G on V by $gv := \mathfrak{F}(g)v$. This induces an action of G on V, since the matrices $\mathfrak{F}(g)$ permute the vectors $v \in V$. Note here that the degree of $\mathfrak{F}$ corresponds to the degree of the module it induces. This turns V into a $\mathbb{C}[G]$-module.

(b) On the other hand, suppose that V is a $\mathbb{C}[G]$-module. Then, V is a $\mathbb{C}$-vector space, so picking some basis $v_1, \ldots, v_n$ of V, we have an action of G on this basis. If we consider an isomorphism $V \longrightarrow \mathbb{C}^n$ sending $v_i \mapsto e_i$, then the action of G on $v_1, \ldots, v_n$ induces an action of G on $\mathbb{C}^n$. Since V is a $\mathbb{C}[G]$-module, then we also expect for all $v, w \in \mathbb{C}^n$ and $c \in \mathbb{C}$ that $g(v+w) = gv + gw$ and $g(cv) = c(gv)$. Therefore, g induces a linear map on $\mathbb{C}^n$, giving a natural matrix representation $\psi(g)$ of each $g \in G$. This gives a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ by $g \mapsto \psi(g)$.

Summing up, the actions of G on an n-dimensional $\mathbb{C}$-vector space $V-$ that is, the structure of a particular $\mathbb{C}[G]$-module – determine a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. On the other hand, a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ determines an action of G on a vector space V, and thus a $\mathbb{C}[G]$-module by extension. We will naturally move back and forth between perspective (a) and perspective (b) in our study of the representation theory of finite groups. In some sources, representations are simply considered to be the actions of G on some $\mathbb{C}[G]$-module, and perspective (a) is derived later.

Now that we have drawn an explicit connection between module theory and the theory of representations, we would like to use some of our tools from module theory to better understand representations. We do that now.

Remark 3.3. We claim that our previous knowledge about $\mathbb{C}[G]$-modules allows us to clearly analyze the structure of a given representation. So, suppose that we consider a representation $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. By Remark 3.2, G acts on some vector space V, so V is a $\mathbb{C}[G]$-module decomposing into $\bigoplus_{i=1}^m N_i$ for irreducible modules $N_1, \ldots, N_m$, and the matrices $\mathfrak{F}(g)$ act on $\bigoplus_{i=1}^m N_i$. For $g \in G$, consider the given matrix map $\mathfrak{F}(g) : \bigoplus_{i=1}^m N_i \longrightarrow \bigoplus_{i=1}^m N_i$. Construct a basis $B = \{v_1^{(1)}, \ldots, v_{k_1}^{(1)}, v_1^{(2)}, \ldots, v_{k_2}^{(2)}, \ldots, v_1^{(n)}, \ldots, v_{k_n}^{(n)}\}$, so that each set $v_1^{(i)}, \ldots, v_{k_i}^{(i)}$ is a basis for N_i. Then, $\mathfrak{F}(g)$ linearly permutes the vectors of $\mathrm{Span}\{v_1^{(i)}, \ldots, v_{k_i}^{(i)}\}$, so the matrix $\mathfrak{F}(g)$ with respect to the basis B decomposes into a block diagonal form, with each block corresponding to a given irreducible module on the basis $v_1^{(i)}, \ldots, v_{k_i}^{(i)}$, and each restriction $\mathfrak{F}|_{N_i}$ corresponding to one block in $\mathfrak{F}(g)$ acting on N_i.

Hopefully, it has now become clear why we spent so long talking about modules and the group algebra. They have powerfully characterized the actions of G on a vector space V. Indeed, we see that studying the representations of G is equivalent to studying the actions of G on the irreducible modules N_i. We now mention a series of definitions and facts, in no particular order, which will be useful to our study of representations and characters.

Definition 3.4. We call a representation $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ *irreducible* if its corresponding $\mathbb{C}[G]$-module is irreducible.

Therefore, each of the restrictions $\mathfrak{F}|_{N_i}$ from Remark 3.3 is an irreducible representation.

Definition 3.5. We say that two representations $\mathfrak{F}, \mathfrak{X} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ are *similar* if there exists a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathfrak{F}(g) = P^{-1}\mathfrak{X}(g)P$ for all $g \in G$.

One may show that similarity is an equivalence relation on the set of irreducible representations.

Lemma 3.6. *Two $\mathbb{C}$-representations $\mathfrak{F}, \mathfrak{X} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ are similar if and only if they induce isomorphic modules.*

Proof. By Remark 3.2, $\mathfrak{F}$ acts on a $\mathbb{C}[G]$-module V and $\mathfrak{X}$ acts on a $\mathbb{C}[G]$-module W. Suppose that there is a $\mathbb{C}[G]$-module isomorphism $\psi : V \longrightarrow W$. Thus, ψ is linear with respect to $\mathbb{C}[G]$, so

$$\psi(\mathfrak{F}(g)v) = \psi(gv) = g\psi(v) = \mathfrak{X}(g)\psi(v),$$

for all $v \in \mathbb{C}^n$. Now, since V, W are both isomorphic to $\mathbb{C}^n$ as $\mathbb{C}$-vector spaces, we may view ψ as a linear $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ automorphism. Therefore, ψ has a corresponding matrix P such that

$$P\mathfrak{F}(g)v = \mathfrak{X}(g)Pv$$

for all v and g. Thus, $\mathfrak{F}(g) = P^{-1}\mathfrak{X}(g)P$ for all $g \in G$.

On the other hand, suppose that $\mathfrak{F} = P^{-1}\mathfrak{X}P$. Once again, take V, W corresponding to $\mathfrak{F}$ and $\mathfrak{X}$. There is a $\mathbb{C}[G]$-isomorphism $\phi : V \longrightarrow \mathbb{C}^n$ mapping $\mathbb{C}$-basis vectors $v_1, \ldots, v_n$ of V to a $\mathbb{C}$-basis $\phi(v_1), \ldots, \phi(v_n)$ for $\mathbb{C}^n$. There is then an automorphism P mapping $\phi(v_1), \ldots, \phi(v_n)$ to $P\phi(v_1), \ldots, P\phi(v_n)$, and finally a $\mathbb{C}[G]$-isomorphism $\psi : \mathbb{C}^n \longrightarrow W$. We now show that this is a $\mathbb{C}[G]$-isomorphism. Of course, each of ϕ, P, ψ is $\mathbb{C}$-linear, so take $g \in G$. Then,

$$\psi(P\phi(gv)) = \psi(P\mathfrak{F}(g)\phi(v)) = \psi(\mathfrak{X}(g)P\phi(v)) = g\psi(P\phi(v)).$$

So, $\psi P \phi$ supplies the necessary $\mathbb{C}[G]$-isomorphism $V \longrightarrow W$. □

Hopefully, Lemma 3.6 has done a good job of motivating our notion of similarity of representations.

Definition 3.7. Let $\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation whose corresponding module is the regular $\mathbb{C}[G]$-module, $\mathbb{C}[G]^\circ$. Then, ρ is called the regular representation.

Example 3.8. The elements of S_3 are $\{1, (1\ 2\ 3), (3\ 2\ 1), (2\ 3), (1\ 2), (1\ 3)\}$, which form a $\mathbb{C}$-basis for $\mathbb{C}[S_3]$. We construct the regular representation $\rho : S_3 \longrightarrow \mathrm{GL}_6(\mathbb{C})$ by analyzing the left action of the generators $(2\ 3), (1\ 2\ 3)$ on $\mathbb{C}[S_3]$. Note that for permutations σ, τ the multiplication $\sigma\tau$ means "do σ, then do τ." Here is a multiplication table.

	1	(1 2 3)	(3 2 1)	(2 3)	(1 2)	(1 3)
(2 3)	(2 3)	(1 2)	(1 3)	1	(1 2 3)	(3 2 1)
(1 2 3)	(1 2 3)	(3 2 1)	1	(1 3)	(2 3)	(1 2)

Therefore, taking $1, (1\ 2\ 3), (3\ 2\ 1), (2\ 3), (1\ 2), (1\ 3)$ to be the ordering of our basis vectors, we can derive the matrices according to the permutations $(1\ 2)$ and $(1\ 2\ 3)$.

$$\rho((2\ 3)) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \rho((1\ 2\ 3)) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

These generate a matrix representation of all $g \in S_3$, and, by extension, give a representation for $\mathbb{C}[S_3]$.

Definition 3.9. Let $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a $\mathbb{C}$-representation of G. Let N be a submodule of the $\mathbb{C}[G]$-module corresponding to $\mathfrak{F}$. Then, the restriction $\mathfrak{F}|_N$ is known as a subrepresentation of $\mathfrak{F}$.

Proposition 3.10. *Every irreducible $\mathbb{C}$-representation is similar to a subrepresentation of the regular representation ρ.*

Proof. Suppose that $\mathfrak{F}$ is an irreducible representation. Then, $\mathfrak{F}$ induces an irreducible $\mathbb{C}[G]$-module over a vector space V. By Proposition 2.18, this irreducible module is isomorphic to a submodule W of $\mathbb{C}[G]^\circ$, and thus $\rho|_W$ is a subrepresentation ρ. Since $W \cong V$, then by Lemma 3.6, $\rho|_W$ and $\mathfrak{F}$ are similar representations. □

So, to study all irreducible $\mathbb{C}$-representations of G, it is sufficient to study the regular representation.

Now that we have finished discussing the regular representation, we mention a few facts about representations in no particular order. We will need these facts at various points throughout the remainder of the thesis.

Remark 3.11. If we consider a representation $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, then $\mathfrak{F}$ extends to a whole map from $\mathbb{C}[G]$ to $M_n(\mathbb{C})$, the ring of $\mathbb{C}$-matrices, by $\sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g \mathfrak{F}(g)$. On the other hand, if we have a map $\mathbb{C}[G] \longrightarrow M_n(\mathbb{C})$, then we may restrict this map to the elements G of $\mathbb{C}[G]$ to get a representation of G. We will not distinguish between these two different viewpoints in general, and freely take representations $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ to also be maps $\mathbb{C}[G] \longrightarrow M_n(\mathbb{C})$ and vice versa.

Lemma 3.12. *Suppose that $\mathfrak{F}_i$ is an irreducible representation corresponding to some irreducible module N_i, and that e_j is the N_j component of 1 for N_j an irreducible $F[G]$-module. Then, if $i \neq j$, we have $\mathfrak{F}_i(e_j) = 0$. Otherwise, $\mathfrak{F}_i(e_i) = I$.*

Proof. So, $\mathfrak{F}_i$ acts on a vector $\mathbb{C}$-vector space V, turning V into a $\mathbb{C}[G]$-module. Considering $e_j \in \mathbb{C}[G]$, if $i \neq j$, then we expect for all $a \in N_i$ that $e_j a = 0$ by Remark 2.32. Thus, for all $v \in V$, we must have $\mathfrak{F}_i(e_j)v = 0$, since $V \cong N_i$ as a $\mathbb{C}[G]$-module. Since this holds for all v, then we conclude that $\mathfrak{F}_i(e_j) = 0$ whenever $i \neq j$. Otherwise $\mathfrak{F}_i(e_i) = I$, since $e_i a = a$ for all $a \in N_i$, and thus $\mathfrak{F}_i(e_i)v = v$ for all $v \in V$. □

For the next Lemma, we first require a definition.

Definition 3.13. The *principal representation* is the representation $\mathfrak{F} : G \longrightarrow M_1(\mathbb{C})$ sending each g in G to 1.

Lemma 3.14. *Let $\mathfrak{F}$ be an irreducible representation. Then, $\mathfrak{F}(G) = 0$ if $\mathfrak{F}$ is not the principal representation.*

Proof. The representation $\mathfrak{F}$ induces an irreducible module on the column space F^n. So, picking $v \in F^n$, let $\mathfrak{F}(G)v = w$. Taking arbitrary $h \in G$,

$$\mathfrak{F}(h)w = \mathfrak{F}(h)\mathfrak{F}(G)v = \mathfrak{F}(hG)v = \mathfrak{F}(G)v = w.$$

So, w is invariant under $\mathfrak{F}$. In particular, the space $\operatorname{Span}\{w\}$ is invariant under the action of $\mathfrak{F}$, and thus $\operatorname{Span}\{w\}$ is an $F[G]$-submodule of F^n. Since F^n is an irreducible submodule, we either have $\operatorname{Span}\{w\} = \{0\}$ or $\operatorname{Span}\{w\} = F^n$. If there is some w such that $\operatorname{Span}\{w\} = F^n$, then for every vector $v \in F^n$, and for all $g \in G$, we have $\mathfrak{F}(g)v = v$, in which case $\mathfrak{F}$ is the principal representation. Otherwise, for all $\mathfrak{F}(G)v = w$, we must have $w = 0$, and thus $\mathfrak{F}(G) = 0$. □

We now prove an important fact about the order of group elements g and their corresponding matrix $\mathfrak{F}(g)$.

Lemma 3.15. *Let $g \in G$ with $|g| = n$, and let $\mathfrak{X}$ be a $\mathbb{C}[G]$-representation acting on N. Then $\mathfrak{X}(g)$ is similar to a diagonal matrix of nth roots of unity.*

Proof. By Remark 3.3, $\mathfrak{X}(g)$ has a block decomposition into irreducible subrepresentations $\mathfrak{F}$ acting on $N_{\mathfrak{F}}$. Since $\mathbb{C}$ is algebraically closed, $\mathfrak{F}$ has an eigenvector x with eigenvalue λ. Moreover, since g has order n, then $I = \mathfrak{F}(1) = \mathfrak{F}(g^n) = \mathfrak{F}(g)^n$. Therefore,

$$x = Ix = \mathfrak{F}(g)^n x = \lambda^n x.$$

So, $\lambda^n = 1$.

Now, consider the eigenspace $E \subseteq N_{\mathfrak{F}}$ of $\mathfrak{F}(g)$. The space E is nonempty since $\mathbb{C}$ is algebraically closed, and $N_{\mathfrak{F}}$ is irreducible. Thus, $E = N_{\mathfrak{F}}$, so $\dim_{\mathbb{C}} E = \dim_{\mathbb{C}} N_{\mathfrak{F}}$. Therefore, $\mathfrak{F}(g)$ must have $\dim_{\mathbb{C}} N_{\mathfrak{F}}$ distinct eigenvectors. Since $\mathfrak{F}$ has dimension $\dim_{\mathbb{C}} N_{\mathfrak{F}}$, and its eigenspace contains $\dim_{\mathbb{C}} N_{\mathfrak{F}}$ distinct eigenvectors, then $\mathfrak{F}$ is diagonalizable with nth roots of unity on the diagonal. This holds for each $\mathfrak{F}$ in the block decomposition of $\mathfrak{X}$, so $\mathfrak{X}$ is also similar to a diagonal matrix of nth roots of unity. □

Remark 3.16. We mention one final fact about the connection between the representations of a group G and the representations of subgroups and quotient groups of G. Essentially, a representation on G gives a representation on its subgroups and quotient groups. Indeed if H is a subgroup of G, and $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(F)$, then the restriction $\mathfrak{F}|_H : H \longrightarrow \mathrm{GL}_n(F)$ is also a representation of H.

Now, consider a quotient group G/N for $N \trianglelefteq G$. Say that $\mathfrak{F}$ is a representation of G, and that $N \trianglelefteq G$ is contained in $\ker \mathfrak{F}$. Then, there is a representation $\widehat{\mathfrak{F}}$ of G/N by $\widehat{\mathfrak{F}}(Ng) := \mathfrak{F}(g)$. This is indeed a representation, for

$$\widehat{\mathfrak{F}}(Ngh) = \mathfrak{F}(gh) = \mathfrak{F}(g)\mathfrak{F}(h) = \widehat{\mathfrak{F}}(Ng)\widehat{\mathfrak{F}}(Nh).$$

On the other hand, if $\widehat{\mathfrak{F}}$ is a representation of G/N, then it is induces a representation of G, by identifying $\widehat{\mathfrak{F}}$ with the map $\widehat{\mathfrak{F}} \circ \pi : G \longrightarrow G/N \longrightarrow \mathrm{GL}_n(\mathbb{C})$ for π the projection $\pi : G \longrightarrow G/N$.

3.2 Characters of Representations

In general, totally classifying even the irreducible representations of a group G is rather challenging. The analysis of something known as the character of a representation makes things substantially easier. As usual, we work over $\mathbb{C}$.

Definition 3.17. Let $\mathfrak{F}$ be a $\mathbb{C}$-representation of G. Then, the *character* $\chi : G \longrightarrow \mathbb{C}$ afforded by $\mathfrak{F}$ is the map $\chi(g) = \mathrm{Tr}(\mathfrak{F}(g))$.

We often apply descriptions of $\mathfrak{F}$ to descriptions of χ. In particular, if $\mathfrak{F}$ has degree n, then we say as well that χ has degree n. Note, in this case, $\chi(1) = n$. Moreover, as mentioned with representations in Remark 3.11, whenever we have a character $\chi : G \longrightarrow \mathbb{C}$, we can extend this to a map $\chi : \mathbb{C}[G] \longrightarrow \mathbb{C}$ by doing $\chi(\sum_{g\in G} c_g g) = \sum_{g\in G} c_g \chi(g)$ for $c_g \in \mathbb{C}$. This extension corresponds to the extension of the representation $\mathfrak{F}$ to a map over all of $\mathbb{C}[G]$, since $\mathrm{Tr}(c\mathfrak{F}(g)) = c\mathrm{Tr}(\mathfrak{F}(g))$.

In general, characters are not group homomorphisms, or anything of that sort. They do however, encode certain properties of their group, and they tell us a great deal about the representations they evaluate. In particular, characters do not distinguish between similar representations, and instead correspond to the isomorphism class of the underlying module.

Lemma 3.18. *The characters of similar $\mathbb{C}$-representations are equal.*

Proof. Say that $\mathfrak{F} = P^{-1}\mathfrak{X}P$, then for all $g \in G$, $\mathrm{Tr}(\mathfrak{F}(g)) = \mathrm{Tr}(P^{-1}\mathfrak{X}(g)P) = \mathrm{Tr}(\mathfrak{X}(g))$, by the cycling property of the trace. □

Lemma 3.19. *For every character χ, $\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof. Suppose that $\mathfrak{F}$ affords χ, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathfrak{F}(g)$. Then, $\lambda_1^{-1}, ..., \lambda_n^{-1}$ are eigenvalues of $\mathfrak{F}(g^{-1}) = \mathfrak{F}(g)^{-1}$. By Lemma 3.15, $\mathfrak{F}(g^{-1})$ has a full set of distinct eigenvalues. Therefore, $\chi(g^{-1}) = \mathrm{Tr}(\mathfrak{F}(g)^{-1}) = \lambda_1^{-1} + ... + \lambda_n^{-1}$. Express λ_i as $a + bi$. Then,

$$\lambda_i^{-1} = (a+bi)^{-1} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

Again by Lemma 3.15, λ_i is an nth root of unity, so $a^2+b^2 = 1$. Therefore, $\lambda_i^{-1} = a-bi = \overline{a+bi} = \overline{\lambda_i}$. Thus,

$$\chi(g^{-1}) = \lambda_1^{-1} + ... + \lambda_n^{-1} = \overline{\lambda_1} + ... + \overline{\lambda_n} = \overline{\lambda_1 + ... + \lambda_n} = \overline{\chi(g)},$$

proving the statement. □

Characters are useful, however, for more than just representations. They form the building blocks for objects known as class functions, which are essential in studying groups.

Definition 3.20. A *class function* $\phi : G \longrightarrow \mathbb{C}$ is a function which is constant over conjugacy classes.

Lemma 3.21. *All characters are class functions.*

Proof. Let $\mathfrak{F}$ afford χ, and suppose that $g = k^{-1}hk$. Then,

$$\chi(g) = \mathrm{Tr}(\mathfrak{F}(g)) = \mathrm{Tr}(\mathfrak{F}(k^{-1}hk)) = \mathrm{Tr}(\mathfrak{F}(k^{-1})\mathfrak{F}(h)\mathfrak{F}(k)) = \mathrm{Tr}(\mathfrak{F}(h)) = \chi(h),$$

again by the cycling property of the trace. $\square$

Remark 3.22. The set of class functions over $\mathbb{C}$ forms a $\mathbb{C}$-vector space. Indeed, suppose that $\phi, \psi : G \longrightarrow \mathbb{C}$ are class functions. Then $\phi + \psi$ is a class function and for all $c \in \mathbb{C}$, $c\phi$ is also a class function.

Definition 3.23. Let $\mathcal{C}(G)$ be the $\mathbb{C}$-vector space of class functions on G.

Definition 3.24. An *irreducible character* is the character of an irreducible representation. For a given group G, we call its set of irreducible characters $\mathrm{Irr}(G)$.

When G is a finite group, $\mathrm{Irr}(G)$ will be finite, since there are only a finite number of possible irreducible representations $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$. Thus, we enumerate the irreducible characters as $\chi_1, \ldots, \chi_n$. Note, that we have not yet proven that the $\chi_1, \ldots, \chi_n$ are all distinct. We do this now. This will come later, but for now any enumeration of characters $\chi_1, \ldots, \chi_n$ should be taken to be the full set of (possibly non-distinct) irreducible characters corresponding to the $\mathfrak{F}_i$. One extremely important property of the irreducible characters is that they form a $\mathbb{C}$-basis for the vector space of class functions, which we will prove in the next section.

After we have introduced the idea of an irreducible character, the reader may not be surprised that we next discuss the notion of the regular character. As with representations and modules, the regular character G is closely intertwined with the irreducible characters of G.

Definition 3.25. The *regular character* χ_ρ is the character afforded by the left regular representation ρ of G.

Lemma 3.26. $\chi_\rho = \sum_{i=1}^{n} \chi_i(1)\chi_i$.

Proof. By Remark 3.3, $\rho(g)$ is a matrix in block diagonal form such that each block corresponds to an irreducible representation of $\mathbb{C}[G]$. Therefore, $\chi_\rho(g)$ is a sum of the traces $\mathrm{Tr}(\rho(g)|_{N_i}) = \chi_i(g)$ of the irreducible representations for each irreducible module N_i of $\mathbb{C}[G]$. By Proposition 2.31, there are exactly $\dim N_i$ subrepresentations of ρ similar to $\rho(g)|_{N_i}$. As mentioned in Remark 3.2, the degree of $\rho(g)|_{N_i}$ is exactly $\dim N_i$, and this degree is $\chi_i(1)$. Therefore, $\chi_\rho = \sum_{i=1}^{n} \chi_i(1)\chi_i$. $\square$

So, as promised, χ_ρ is closely linked to the irreducible characters.

Lemma 3.27. *For all $g \in G$ satisfying $g \neq 1$, $\chi_\rho(g) = 0$. Otherwise, $\chi_\rho(1) = |G|$.*

Proof. The regular representation ρ acts on a space of dimension $|G|$, so its degree is $|G|$, and thus $\chi_\rho(1) = |G|$. Now, consider $\rho(g)$ with respect to the basis $g_1, \ldots, g_n$ of all group elements of G. Then, if $\rho(g)$ has a nonzero diagonal entry, there is some group element $h \in G$, such that gh is fixed. Of course, this occurs only if $g = 1$, contradicting the claim that $g \neq 1$, so all diagonal entries of $\rho(g)$ are 0, and thus $\chi_\rho(g) = 0$ whenever $g \neq 1$. $\square$

Lemma 3.26 and Lemma 3.27 already provide one of our first interesting applications of characters.

Proposition 3.28. $|G| = \sum_{i=1}^{n} \chi_i(1)^2$.

Proof. By Lemma 3.26, $\chi_\rho(1) = \sum_{i=1}^{n} \chi_i(1)^2$, and by Lemma 3.27, $\chi_\rho(1) = |G|$. □

Example 3.29. This proposition already tells us some useful things. Consider, for example, the case $\mathbb{C}[C_3]$. Since C_3 is an order 3 group, and every integer $a > 1$ has a square strictly larger than 3, then each irreducible character of C_3 has degree 1, and there are exactly 3 irreducible characters, and thus irreducible representations, of C_3.

3.3 The First Orthogonality Relation

We will now use Lemma 3.26 and Lemma 3.27 to extract something known as the First Orthogonality Relation on the irreducible $\mathbb{C}$-characters. This will prove to be enormously useful, and it is exactly the inner product on class functions which we promised earlier. The reader beware that this entire section follows [1, Chapter 2] especially closely. We will need the this piece of notation for several of the following lemmas.

Definition 3.30. We define the term δ_{ij} by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This is known as the *Kronecker delta.*

I cannot supply any inspired piece of wisdom for why the following two results hold. They are the result of a few rather technical and clever computations, which I have only found in [1, Chapter 2]. It may be helpful to review Remark 2.32 to recall the elements e_i of $\mathbb{C}[G]$. The reader may compare this next Lemma 3.31 to [1, Theorem 2.12].

Lemma 3.31. *Let e_i be the N_i component of 1 in the decomposition $\mathbb{C}[G] = \bigoplus_{i=1}^{k} N_i$. Then,*

$$e_i = \frac{1}{|G|} \left(\sum_{g \in G} \chi_i(1)\chi_i(g^{-1})g \right).$$

Proof. Recall that each e_i is the 1 component of N_i. Consider once again the $\mathbb{C}$-basis of group elements $g \in G$. Thus, for some $c_h \in \mathbb{C}$, we have $e_i = \sum_{h \in G} c_h h$, for $c_h \in \mathbb{C}$. We use Lemma 3.27 to evaluate e_i multiplied by arbitrary g^{-1} under χ_ρ. So,

$$\chi_\rho(e_i g^{-1}) = \chi_\rho\left(\left(\sum_{h \in G} c_i h \right) g^{-1} \right) = \chi_\rho(c_g g g^{-1}) + \sum_{h \in G,\, h \neq g} \chi_\rho(c_h h g^{-1}) = \chi_\rho(c_g 1) = c_g \chi_\rho(1),$$

where, since $hg^{-1} \neq 1$, we have

$$\sum_{h \in G,\, h \neq g} \chi_\rho(c_h h g^{-1}) = \sum_{h \in G,\, h \neq g} c_h \chi_\rho(hg^{-1}) = \sum_{h \in G,\, h \neq g} 0 = 0$$

by Lemma 3.27. Again, using Lemma 3.27, $c_g\chi_\rho(1) = c_g|G|$, so $\chi_\rho(e_ig^{-1}) = c_g|G|$.

We now calculate $\chi_\rho(e_ig^{-1})$ in a slightly different way. Let $\mathfrak{F}_j$ be the representation affording χ_j. Recall by Remark 3.11 that we may extend $\mathfrak{F}_j$ to all of $\mathbb{C}[G]$. Moreover, as in Proposition 3.12, $\mathfrak{F}_j(e_i)$ is the zero matrix whenever $i \neq j$. Otherwise, $\mathfrak{F}_i(e_i) = I$. Therefore,

$$\chi_j(e_ig^{-1}) = \mathrm{Tr}(\mathfrak{F}_j(e_ig^{-1})) = \mathrm{Tr}(\mathfrak{F}_j(e_i)\psi_j(g^{-1})) = \delta_{ij}\chi_j(g^{-1}),$$

then

$$\sum_j \chi_j(1)\chi_j(e_ig^{-1}) = \sum_j \delta_{ij}\chi_j(1)\chi_j(g^{-1}) = \chi_i(1)\chi_i(g^{-1})$$

Thus, $c_g|G| = \chi_j(e_ig^{-1}) = \chi_i(1)\chi_i(g^{-1})$, so $c_g = \frac{1}{|G|}\chi_i(1)\chi_i(g^{-1})$, and

$$e_i = \sum_{g\in G} c_g g = \sum_{g\in G} \frac{1}{|G|}\chi_i(1)\chi_i(g^{-1})g = \frac{1}{|G|}\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})g.$$

So, the claim is proven. □

Note that we have also just shown that χ_i, χ_j are distinct characters whenever they correspond to distinct irreducible representations $\mathfrak{F}_i, \mathfrak{F}_j$, for $\chi_i(e_j) = \mathrm{Tr}(\mathfrak{F}_i(e_j)) = 0$, but $\chi_j(e_j) = \mathrm{Tr}(\mathfrak{F}(e_j)) = \mathrm{Tr}(I) \neq 0$. What follows is called the Generalized Orthogonality Relation. Compare Theorem 3.32 to [1, Theorem 2.13].

Theorem 3.32 (Generalized Orthogonality Relation)**.** *For all $h \in G$ and $\chi_i, \chi_j \in \mathrm{Irr}(G)$,*

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(gh)\overline{\chi_j(g)} = \delta_{ij}\frac{\chi_i(h)}{\chi_j(1)}$$

Proof. Note by Remark 2.32 that for all e_i, e_j, we have $e_ie_j = \delta_{ij}e_i$. On the other hand, by Lemma 3.31,

$$e_ie_j = \frac{1}{|G|^2}\left(\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})g\right)\left(\sum_{g\in G}\chi_j(1)\chi_j(g^{-1})g\right).$$

We simplify the right hand side of the above equation and switch the order of the summation. So,

$$\frac{1}{|G|^2}\left(\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})g\right)\left(\sum_{h\in G}\chi_j(1)\chi_j(h^{-1})h\right) = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{h\in G}\sum_{g\in G}\chi_i(g^{-1})\chi_j(h^{-1})gh$$

Ranging over $g \in G$ is identical to ranging over k for $kh^{-1} \in G$ with h fixed. Thus,

$$\frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{h\in G}\sum_{g\in G}\chi_i(g^{-1})\chi_j(h^{-1})gh = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{h\in G}\sum_{k\in G}\chi_i((kh^{-1})^{-1})\chi_j(h^{-1})(kh^{-1})h$$

We simplify and then swap the summation order one final time, and set this equation equal to $\delta_{ij}e_i$ which is $\frac{\delta_{ij}}{|G|}\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})g$ by Lemma 3.31.

$$\frac{\delta_{ij}}{|G|}\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})g = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{k\in G}\left(\sum_{h\in G}\chi_i((kh^{-1})^{-1})\chi_j(h^{-1})\right)k.$$

Since the elements $g \in G$ form a basis for $\mathbb{C}[G]$, we compare the above equivalence for a particular $k \in G$. So,

$$\frac{\delta_{ij}}{|G|}\chi_i(1)\chi_i(k^{-1}) = \frac{\chi_i(1)\chi_j(1)}{|G|^2}\sum_{h\in G}\chi_i((kh^{-1})^{-1})\chi_j(h^{-1}).$$

We multiply both sides by $|G|$ and divide both sides by $\chi_i(1)\chi_j(1)$, giving

$$\delta_{ij}\frac{\chi_i(k^{-1})}{\chi_j(1)} = \frac{1}{|G|}\sum_{h\in G}\chi_i(hk^{-1})\chi_j(h^{-1}).$$

We reach our final formulation by setting k^{-1} to be g, and remarking that $\chi_j(h^{-1}) = \overline{\chi_j(h)}$ by Lemma 3.19.

$$\delta_{ij}\frac{\chi_i(k)}{\chi_j(1)} = \frac{1}{|G|}\sum_{h\in G}\chi_i(hg)\overline{\chi_j(h)}.$$

$\square$

Though the generalized Orthogonality Relation is very powerful, it typically is translated into the following more compact and useful form. Once again, compare this next theorem, Theorem 3.33, to [1, Corollary 2.14].

Theorem 3.33 (First Orthogonality Relation)**.**

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij}.$$

Proof. By the Generalized Orthogonality Relation (Theorem 3.32), taking a fixed $g \in G$, then

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(gh)\overline{\chi_j(g)} = \delta_{ij}\frac{\chi_i(h)}{\chi_j(1)}$$

holds for every $h \in G$. Taking $h = 1$, then

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \delta_{ij}\frac{\chi_i(1)}{\chi_j(1)} = \delta_{ij},$$

given that whenever $\delta_{ij} = 1$, then $i = j$, so that $\frac{\chi_i(1)}{\chi_j(1)} = 1$. Otherwise, $\delta_{ij}\frac{\chi_i(1)}{\chi_j(1)} = 0$. $\square$

The reader may note that the First Orthogonality Relation looks extremely similar to something like a standard inner product on $\mathbb{R}$ or $\mathbb{C}$. We claim that this is indeed an inner product, which we justify in the next section.

3.4 An Inner Product and Basis for $\mathcal{C}(G)$

We now define our inner product.

Definition 3.34. Let ϕ, θ be class functions. We define $[\phi, \theta] := \frac{1}{|G|}\sum_{g \in G} \phi(g)\overline{\theta(g)}$ and call this the *inner product* over the space of $\mathcal{C}(G)$ of class functions.

Proposition 3.35. *The inner product over the space $\mathcal{C}(G)$ satisfies the standard definition of an inner product.*

Proof. The inner product $[\cdot, \cdot]$ satisfies the properties we expect of an inner product. Namely,

(a) $[\phi, \theta] = \overline{[\theta, \phi]}$

$$[\phi, \theta] = \frac{1}{|G|}\sum_{g \in G} \phi(g)\overline{\theta(g)} = \overline{\left(\frac{1}{|G|}\sum_{g \in G} \theta(g)\overline{\phi(g)}\right)} = \overline{[\theta, \phi]}$$

(b) Positivity, that is $[\phi, \phi] > 0$ unless $\phi = 0$. Taking $\phi(g) = a_g + b_g i$ for $a, b \in \mathbb{R}$. Then,

$$[\phi, \phi] = \frac{1}{|G|}\sum_{g \in G} \phi(g)\overline{\phi(g)} = \frac{1}{|G|}\sum_{g \in G}(a_g + b_g i)(a_g - b_g i) = \frac{1}{|G|}\sum_{g \in G} a_g^2 + b_g^2 \geqslant 0$$

reaching equality only when $a_g^2 + b_g^2 = 0$ for all a_g, b_g. But this holds only if $a_g, b_g = 0$, in which case $\phi = 0$.

(c) And $[c_1\phi_1 + c_2\phi_2, \theta] = c_1[\phi_1, \theta] + c_2[\phi_2, \theta]$. This is known as linearity in the first term. Indeed, $[c_1\phi_1 + c_2\phi_2, \theta] = c_1[\phi_1, \theta] + c_2[\phi_2, \theta]$, since

$$[c_1\phi_1 + c_2\phi_2, \theta] = \frac{1}{|G|}\sum_{g \in G}(c_1\phi_1 + c_2\phi_2)\overline{\theta(g)} = c_1\frac{1}{|G|}\sum_{g \in G}\phi_1\overline{\theta(g)} + c_2\frac{1}{|G|}\sum_{g \in G}\phi_2\overline{\theta(g)}.$$

Note that by (a), $[\phi, c\theta] = \overline{[c\theta, \phi]} = \overline{c}\overline{[\theta, \phi]} = \overline{c}[\phi, \theta]$. This is known as conjugate linearity in the second term.

$\square$

Thus, restating the First Orthogonality Relation, we have $[\chi_i, \chi_j] = \delta_{ij}$ for all irreducible $\chi_i, \chi_j \in \text{Irr}(G)$. In fact, the irreducible characters are linearly independent in $\mathcal{C}(G)$.

Lemma 3.36. *Enumerate the irreducible representations $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$, and let χ_i be the irreducible character corresponding to $\mathfrak{F}_i$. Then, the χ_i are linearly independent.*

Proof. Consider a sum $\sum_{i=1}^{n} c_i\chi_i$, and suppose that $c_j = 0$. Then, by linearity in the first term of the inner product, we have

$$\left[\sum_{i=1}^{n} c_i\chi_i, \chi_j\right] = \sum_{i=1}^{n} c_i[\chi_i, \chi_j] = \sum_{i=1}^{n} c_i \cdot 0 = 0.$$

So, $\chi_j \neq \sum_{i=1}^{n} c_i\chi_i$. □

We now want to prove that the irreducible characters also form a basis for $\mathcal{C}(G)$. The proof for Theorem 3.37 follows [4, 2.6, Theorem 5] closely.

Theorem 3.37. *The elements of* $\mathrm{Irr}(G)$ *form an orthonormal basis for* $\mathcal{C}(G)$.

Proof. By Lemma 3.36, the irreducible characters are all linearly independent. Moreover, by Theorem 3.33, the linear characters are orthonormal. Now, set $V = \mathrm{Span}\{\chi_1, \ldots, \chi_n\}$ and consider the map $T : \mathcal{C}(G) \longrightarrow V$ by $Tv = v = \sum_{i=1}^{n}[v, \chi_i]\chi_i$. Since the inner product is linear in the first term, then T is a linear map. We want to prove $\ker T = \{0\}$, showing that $\mathcal{C}(G) = V$. This is equivalent to showing that for all $\phi \in \mathcal{C}(G)$, if $[\chi_i, \phi] = 0$, then $\phi = 0$.

Let $\chi \in \mathrm{Irr}(G)$ and pick $\mathfrak{F} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ affording χ. Set $V = \mathbb{C}^n$, and take the induced action of G on V by $gv = \mathfrak{F}(g)v$. Define $\psi_{\mathfrak{F}} : V \longrightarrow V$ by

$$\psi_{\mathfrak{F}}(v) = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\mathfrak{F}(g)(v).$$

This is a $\mathbb{C}[G]$-homomorphism $V \longrightarrow V$. Since $\mathfrak{F}$ is irreducible, V is irreducible, so by Lemma 2.26, $\psi_{\mathfrak{F}} = \lambda \mathrm{id}_V$ for $\lambda \in \mathbb{C}$.

Since $\dim_{\mathbb{C}} V = n$, then $\mathrm{Tr}(\psi_{\mathfrak{F}}) = \lambda \mathrm{Tr}(I_n) = \lambda n$. So,

$$\lambda \cdot n = \mathrm{Tr}(\psi_{\mathfrak{F}}) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\mathfrak{F}(g)\right) = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\chi(g) = 0$$

proving $\lambda = 0$, and thus $\psi_{\mathfrak{F}} = 0$, for all representations $\mathfrak{F}$. Take the case $\mathfrak{F} = \rho$, the regular representation. In this case, $V = \mathbb{C}[G]^\circ$. So, $V \cong \bigoplus_{i=1}^{n} N_i$. Since G permutes the $\mathbb{C}$-basis vectors of V faithfully, and $|G|$ equals the number of $\mathbb{C}$-basis vectors in V, then for $v \in V$, we have $\rho(g)v \neq \rho(h)v$ for all $h, g \in G$. Take all $1 = g_0, \ldots, g_r \in G$, and the corresponding basis $v_{g_0}, \ldots, v_{g_r}$ such that $\rho(g_i)v_1 = v_{g_i}$. Then,

$$\psi_\rho(v_1) = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}\rho(g)(v_1) = \frac{1}{|G|}\sum_{g\in G} \overline{\phi(g)}v_g = 0.$$

Since the v_g are linearly independent, then $\overline{\phi(g)} = 0$. So, $\phi(g) = 0$ over all g, proving $\phi = 0$. Therefore, $\ker T = \{0\}$, so $T : \mathcal{C}(G) \longrightarrow V$ is an isomorphism, proving $V = \mathcal{C}(G)$. □

We now prove a collection of results which all follow from Theorem 3.37.

Corollary 3.38. *The number of similarity classes of irreducible representations equals the number of conjugacy classes in* G.

Proof. By Theorem 3.37, the characters $\chi \in \mathrm{Irr}(G)$ form a basis for $\mathcal{C}(G)$. Alternatively, every class function is constant over the $h_1^G, ..., h_k^G$ conjugacy classes of G, so every class function is determined by its values $\lambda_1, ..., \lambda_k$ over the conjugacy classes, which may be chosen arbitrarily. So the dimension of the space of class functions equals k. Thus, $|\mathrm{Irr}(G)| = k$. □

Corollary 3.39. *Let ϕ be a class function. Then, $\phi = \sum_{i=1}^{n}[\phi, \chi_i]\chi_i$. Moreover, if ϕ is a character, then the coefficients $[\phi, \chi_i]$ are nonnegative integers, and ϕ is irreducible if and only if $[\phi, \phi] = 1$.*

Proof. Consider again the isomorphism T from Theorem 3.37. In particular, T is injective, so there are no two distinct class functions ϕ, ψ such that $T\phi = T\psi$. Therefore, there are no two distinct class functions ϕ, ψ such that $[\chi_i, \phi] = [\chi_i, \psi]$ for all i. Thus, for all j with $1 \leqslant j \leqslant n$, we have

$$\left[\sum_{i=1}^{n}[\phi, \chi_i]\chi_i, \chi_j\right] = \sum_{i=1}^{n}[\phi, \chi_i][\chi_i, \chi_j] = [\phi, \chi_j][\chi_j, \chi_j] = [\phi, \chi_j].$$

Therefore, $\sum_{i=1}^{n}[\phi, \chi_i]\chi_i = \phi$.

Now, if ϕ is a character, then the representation $\mathfrak{X}$ affording ϕ has a block decomposition into irreducible representations. Therefore, a given irreducible representation $\mathfrak{F}$ appears in the decomposition of $\mathfrak{X}$ an integer number of times, and so $\phi = \sum_{i=1}^{n} c_i\chi_i$ for c_i a nonnegative integer. Since $\phi = \sum_{i=1}^{n}[\phi, \chi_i]\chi_i$ as well, then $[\phi, \chi_i] = c_i \in \mathbb{Z}^{\geqslant 0}$. Moreover, if $[\phi, \phi] = 1$, then $1 = \sum_{i=1}^{n}[\phi, \chi_i]^2[\chi_i, \chi_i]^2 = \sum_{i=1}^{n}[\phi, \chi_i]^2$. Therefore, there is one χ_i such that $[\phi, \chi_i] = 1$, so $\phi = \chi_i$. □

Lemma 3.40. *Two representations are similar if and only if they afford equal characters.*

Proof. If two representations are isomorphic, then we have already shown that they afford equal characters. Alternatively, suppose two representations $\mathfrak{F}, \mathfrak{X}$ afford equal characters, ϕ, θ. Then $\phi = \sum_{i=1}^{n}[\phi, \chi_i]\chi_i = \theta$ by Corollary 3.39. Since χ_i is irreducible, there is no sum of characters equivalent to χ_i. Therefore, the irreducible representation corresponding to χ_i must appear $[\phi, \chi_i]$ times in the block decomposition of $\mathfrak{F}$ and $\mathfrak{X}$. Therefore, $\mathfrak{F}$ and $\mathfrak{X}$ have identical block decompositions, so they are similar by Lemma 3.6. □

For this next theorem, we denote the centralizer subgroup of $g \in G$ by the notation $C_G(g)$. Compare Theorem 3.41 to [1, Theorem 2.18].

Theorem 3.41 (Second Orthogonality Relation)**.** *Pick a set of representatives $h_1, ..., h_k$ of conjugacy classes in G. For representatives $h_n, h_m \in G$,*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(h_n)\overline{\chi(h_m)} = |C_G(h_n)|\delta_{nm}$$

Proof. List all irreducible characters $\chi_1, ..., \chi_k$ of G and pick a list $h_1, ..., h_k$ of representatives of each conjugacy class in G. By Corollary 3.38, we know these lists are the same length, thus we may make a square matrix

$$P = \begin{pmatrix} \chi_1(h_1) & \chi_1(h_2) & \dots & \chi_1(h_k) \\ \chi_2(h_1) & \ddots & & \chi_2(h_k) \\ \vdots & & \ddots & \vdots \\ \chi_k(h_1) & \chi_k(h_2) & \dots & \chi_k(h_k) \end{pmatrix}.$$

By the First Orthogonality Relation, we have

$$|G|\delta_{ij} = \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)}.$$

Since each $\chi_i(g)\overline{\chi_j(g)}$ is constant over conjugacy classes, then

$$|G|\delta_{ij} = \sum_{i=1}^{k} |h_n^G|\chi_i(h_n)\overline{\chi_j(h_n)}$$

Denote by $\overline{P}$ the matrix of elements conjugate to the elements of P. Then, entry a_{ij} of the matrix $P\overline{P^T}$, is $\sum_{n=1}^{k} \chi_i(h_n)\overline{\chi_j(h_n)}$. So we define

$$D = \begin{pmatrix} |h_1^G| & & \\ & \ddots & \\ & & |h_k^G| \end{pmatrix}$$

where the entries of $PD\overline{P^T}$ are $a_{ij} = \sum_{n=1}^{k} |h_i^G|\chi_i(h_n)\overline{\chi_j(h_n)}$. Again by the First Orthogonality Relation, $a_{ij} = |G|\delta_{ij}$. Thus, $PD\overline{P^T} = |G|I$, so $D\overline{P^T}$ is a scalar multiple the inverse of P. Thus, $(DP^T)P = |G|I$. We multiply both sides by D^{-1}, giving $\overline{P^T}P = |G|D^{-1}$.

$$\begin{pmatrix} \overline{\chi_1(h_1)} & \dots & \overline{\chi_k(h_1)} \\ \vdots & & \vdots \\ \overline{\chi_1(h_k)} & \dots & \overline{\chi_k(h_k)} \end{pmatrix} \begin{pmatrix} \chi_1(h_1) & \dots & \chi_1(h_k) \\ \vdots & & \vdots \\ \chi_k(h_1) & \dots & \chi_k(h_k) \end{pmatrix} = \begin{pmatrix} \frac{|G|}{|h_1^G|} & & \\ & \ddots & \\ & & \frac{|G|}{|h_k^G|} \end{pmatrix}$$

Multiplying out each entry a_{nm} gives

$$a_{nm} = \sum_{\chi \in \mathrm{Irr}(G)}^{k} \chi(h_n)\overline{\chi(h_m)} = \frac{\delta_{nm}|G|}{|h_n^G|} = |C_G(h_n)|\delta_{nm}$$

for $C_G(h_n)$ the centralizer of h_n. $\square$

Considering again the matrix P from Theorem 3.41, then we may think of the First Orthogonality Relation as row orthogonality of P, and the Second Orthogonality Relation as column orthogonality of P.

3.5 Properties of Characters

Now that we have described the structure of the space $\mathcal{C}(G)$, we may move on to studying characters themselves. What follows is a somewhat random collection of properties of characters, and certain special types of characters. We only assemble the results necessary for the remainder of the thesis. For further information, we refer the reader to [1, Chapter 2].

Lemma 3.42. *Let $\mathfrak{F}$ be a $\mathbb{C}$-representation of G affording χ. Then, $g \in \ker \mathfrak{F}$ if and only if $\chi(g) = \chi(1)$.*

Proof. Suppose that $g \in \ker \mathfrak{F}$. Then, $\mathfrak{F}(g) = \mathfrak{F}(1)$, and so $\chi(g) = \chi(1)$. Now, suppose that $\chi(g) = \chi(1)$. Then, by Lemma 3.15, $\chi(g) = \zeta_1 + \cdots + \zeta_n$ for $n = \chi(1)$ and $\zeta_1, \dots, \zeta_n$ distinct $|g|$th roots of unity. Given that $|\zeta_i| = 1$, then $\chi(g) = \chi(1)$ forces $\zeta_i = 1$ for each i. Thus, $\mathfrak{F}(g)$ is similar to I_n, again by Lemma 3.15, so $\mathfrak{F}(g) = I_n$, and thus $g \in \ker \mathfrak{F}$. □

Definition 3.43. Let χ be an $\mathbb{C}$-character of G. We define $\ker \chi := \{g \in G : \chi(g) = \chi(1)\}$.

Thus, the previous Lemma states that $\ker \chi = \ker \mathfrak{F}$ for $\mathfrak{F}$ affording χ.

Lemma 3.44. *Let $N \trianglelefteq G$. Then,*

(a) If χ is a character of G with $N \subseteq \ker \chi$, then for $g_1, \dots, g_k$ a distinct set of coset representatives of N, χ is constant on all $h \in Ng_i$ for each g_i. Moreover, the function $\hat{\chi} : G/N \longrightarrow \mathbb{C}$ defined by $\hat{\chi}(Ng_i) = \chi(g_i)$ is a character on G/N.

(b) If $\hat{\chi}$ is a character of G/N, then χ defined by $\chi(g) := \hat{\chi}(Ng)$ is a character of G.

(c) In (a) and (b), we have $\chi \in \operatorname{Irr}(G)$ if and only if $\hat{\chi} \in \operatorname{Irr}(G/N)$.

Proof. Both (a) and (b) follow as a result of Remark 3.16. It remains to prove (c). As before, let $g_1, \dots, g_k$ be coset representatives of N. Suppose $\hat{\chi}(Ng) = \chi(g)$. Then, we have the following equality,

$$\frac{1}{|G|}\sum_{g \in G} \chi(g)\overline{\chi(g)} = \frac{1}{|G|}\sum_{i=1}^{k} \chi(Ng_i)\overline{\chi(Ng_i)} = \frac{1}{|G|}\sum_{i=1}^{k} |N|\hat{\chi}(g_i)\overline{\hat{\chi}(g_i)} = \frac{1}{|G/N|}\sum_{Ng \in G/N} \hat{\chi}(Ng)\overline{\hat{\chi}(Ng)}.$$

Thus, $[\chi, \chi] = [\hat{\chi}, \hat{\chi}]$, so χ is irreducible if and only if $\hat{\chi}$ is irreducible. □

Definition 3.45. A *linear character* is a character of degree 1.

One (and perhaps the most important) example of a linear character is known as the principal character (or *trivial character*), $1_G : G \longrightarrow \mathbb{C}$, which is the map $1_G(g) = 1$. Note that this is the character of the principal representation given by Definition 3.13. Since every group G has a homomorphism $G \longrightarrow \{1\}$, then every group has a trivial character.

Lemma 3.46. *Linear characters are group homomorphisms into an abelian subgroup of $\mathbb{C}^\times$.*

Proof. Let $\mathfrak{F}$ afford a linear character ξ. Then,

$$\xi(gh) = \operatorname{Tr}(\mathfrak{F}(gh)) = \operatorname{Tr}(\mathfrak{F}(g))\operatorname{Tr}(\mathfrak{F}(h)) = \xi(g)\xi(h).$$

□

Remark 3.47. Thus all linear characters are in fact representations of G, since $\mathbb{C}^\times$ is a subgroup of $\mathrm{GL}_1(\mathbb{C})$. Since, moreover, each representation of G has nth roots of unity for its eigenvalues, and each linear character is a 1-dimensional representation, then each ξ evaluates to nth roots of unity for some n by Lemma 3.15.

Lemma 3.48. *All linear characters of a group G are irreducible.*

Proof. Let ξ be a linear character. Then, using the fact from Lemma 3.46 that linear characters are homomorphisms, and that $\overline{\xi(g)} = \xi(g^{-1})$ by Lemma 3.19, we have

$$[\xi, \xi] = \frac{1}{|G|}\sum_{g \in G} \xi(g)\overline{\xi(g)} = \frac{1}{|G|}\sum_{g\in G}\xi(g)\xi(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\xi(gg^{-1}) = \frac{1}{|G|}|G| = 1.$$

So, ξ is irreducible by Lemma 3.39. □

Lemma 3.49. *A group G is abelian if and only if all of its characters are linear.*

Proof. If all characters of G are linear, then, by Proposition 3.28, there are $|G|$ irreducible characters of G, so by Corollary 3.38 there are $|G|$ conjugacy classes of G, and thus G is abelian. Alternatively, if G is abelian, then moving in the reverse direction, G has $|G|$ conjugacy classes, and thus $|G|$ characters, so all its characters are degree 1 and thus linear. □

Proposition 3.50. *Let G be an abelian group. Then,* $\mathrm{Irr}(G)$ *forms a group under multiplication defined by $\xi_1\xi_2(g) := \xi_1(g)\xi_2(g)$ for $\xi_1, \xi_2 \in \mathrm{Irr}(G)$ and $g \in G$. Furthermore* $\mathrm{Irr}(G) \cong G$.

Proof. We first show $\mathrm{Irr}(G)$ forms a group. Pick $\xi_1, \xi_2 \in \mathrm{Irr}(G)$. We calculate $[\xi_1\xi_2, \xi_1\xi_2] = 1$. Indeed,

$$\frac{1}{|G|}\sum_{g\in G}\xi_1\xi_2(g)\overline{\xi_1\xi_2(g)} = \frac{1}{|G|}\sum_{g\in G}\xi_1(g)\xi_1(g^{-1})\xi_2(g)\xi_2(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}(1)(1) = 1$$

The principal character is the identity in $\mathrm{Irr}(G)$ and for each $\xi : G \longrightarrow \mathbb{C}$ we may define ξ^{-1} by $\xi^{-1}(g) = \xi(g^{-1})$, which is a class function and a homomorphism. Furthermore, $[\xi^{-1}, \xi^{-1}] = [\xi, \xi] = 1$, so $\xi^{-1} \in \mathrm{Irr}(G)$ by Lemma 3.39, and $\xi^{-1}\xi = 1_G$, since $\xi^{-1}\xi(g) = \xi(g)^{-1}\xi(g)\xi(g^{-1}g) = 1$, so we have constructed an inverse for every ξ. Thus, $\mathrm{Irr}(G)$ is a group.

Now, pick generators $g_1, \ldots, g_r$ of G. Define ξ_i by $\xi_i(g_j) = \zeta$ for ζ a $|g_i|$th root of unity if $i = j$ and $\xi_i(g_j) = 1$ otherwise. For every ξ_i, $[\xi_i, \xi_i] = 1$, so the ξ_i are irreducible linear characters. Moreover, every linear character is some product $\Pi_{i=1}^r \xi_i^{k_i}$ for exponents k_i. The map $\phi : \mathrm{Irr}(G) \longrightarrow G$ by $\xi_i \mapsto g_i$ then gives an isomorphism of groups. □

We now characterize the set of linear characters of a group G, for which we require the following definition.

Definition 3.51. Denote by $[G, G]$ the derived subgroup of G, the group generated by all commutators $[g, h]$ of G.

Proposition 3.52. *Let G be a group, and let $\mathcal{L}$ be the set of linear characters of G. Then $\mathcal{L} \cong G/[G,G]$.*

Proof. By Lemma 3.44, there is a one to one correspondence between the characters of G containing $[G,G]$ in their kernel, and the characters of $G/[G,G]$. Moreover, the linear characters of G are uniquely the set of characters containing $[G,G]$ in their kernel, for any homomorphism of G into an abelian group contains $[G,G]$ in the kernel. Thus, define the map $\widehat{\ } : \mathcal{L} \longrightarrow \mathrm{Irr}(G/[G/G])$ by $\xi \mapsto \hat{\xi}$. Again, using Lemma 3.44, $\hat{\xi}$ is uniquely defined by ξ, and ξ is uniquely defined by $\hat{\xi}$, so this is a bijection. Now,

$$\widehat{\xi_1\xi_2}(Ng) = \xi_1\xi_2(g) = \xi_1(g)\xi_2(g) = \hat{\xi}_1(Ng)\hat{\xi}_2(Ng).$$

Therefore, this is a homomorphism, and thus an isomorphism. Finally, by Proposition 3.50, $\mathrm{Irr}(G/[G,G]) \cong G/[G,G]$, so $\mathcal{L} \cong G/[G,G]$. □

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