**Real Analysis Qual, Fall 2018**

**Problem 1.** Let $f(x) = 1/x$. Show that $f(x)$ is uniformly continuous on $(1,\infty)$ but not on $(0,1)$.

*Proof.* Take $\epsilon > 0$. Choose $M$ large enough such that for all $x \geqslant M$ we have $|f(x)| < \epsilon/4$. Since $f(x)$ decreases monotonically to 0, such an $M$ exists. Then, for all $x, y \geqslant M$, we have $|f(x) - f(y)| \leqslant |f(x)| + |f(y)| \leqslant \epsilon/4 + \epsilon/4 = \epsilon/2$. Now, $f$ is a continuous function on the compact interval $[1, M]$. So, there exists $\delta > 0$ such that for all $x, y \in [1, M]$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/2$. Suppose that $x \in [1, M]$ and $y \in [M, \infty)$ are such that $|x - y| < \delta$. Then,

$$|f(x) - f(y)| \leqslant |f(x) - f(M)| + |f(M) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, $f$ is uniformly continuous on $[1,\infty)$, and therefore uniformly continuous on $(1,\infty)$. On the other hand, for any $\delta > 0$, take $a < \delta$, and consider $x, x + a$. Then $|x - (x + a)| < \delta$. However,

$$\frac{1}{x} - \frac{1}{x+a} = \frac{x+a}{x(x+a)} - \frac{x}{x(x+a)} = \frac{a}{x(x+a)}.$$

Taking $x \to 0$ sends $\frac{a}{x(x+a)} \to \infty$. So, for any $\epsilon$, there is no $\delta$ such that $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$ with $x, y \in (0, 1)$. Thus, $f$ is not uniformly continuous on $(0, 1)$. □

**Problem 2.** Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set. Show that there is a Borel set $B \subseteq E$ such that $m(E \backslash B) = 0$.

**Problem 3.** Suppose $f(x)$ and $xf(x)$ are integrable on $\mathbb{R}$. Define $F$ by

$$F(t) = \int_{-\infty}^{\infty} f(x)\cos(xt)\,\mathrm{d}x.$$

Show that

$$F'(t) = -\int_{-\infty}^{\infty} xf(x)\sin(xt)\,\mathrm{d}x.$$

*Proof.* Set $h(x, t) = f(x)\cos(xt)$. Observe that $|h(x, t)| \leqslant |f(x)\cos(xt)| \leqslant |f(x)| \in L^1$ for all $t$. Moreover,

$$\left|\frac{\partial}{\partial t}h(x,t)\right| = |f(x)(-x\sin(xt))| \leqslant |f(x)x| \in L^1$$

for all $t$. Moreover, $F(t) = \int h(x, t)\,\mathrm{d}x$. These are the necessary conditions for differentiation under the integral sign. In particular, we obtain

$$F'(t) = \int \frac{\partial}{\partial t}h(x,t)\,\mathrm{d}x = \int f(x)(-x\sin(xt))\,\mathrm{d}x = -\int xf(x)\sin(xt)\,\mathrm{d}x.$$

□

**Problem 4.** **(Classic)** Let $f \in L^1([0, 1])$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)|\sin(nx)|\,\mathrm{d}x = \frac{2}{\pi}\int_0^1 f(x)\,\mathrm{d}x.$$

Hint: Begin with the case in which $f$ is the characteristic function of an interval.

*Proof.* We first integrate a single arc of the function $|\sin(nx)|$ along the interval $(0, \pi/n)$. So,

$$\int_0^{\pi/n} |\sin(nx)|\,\mathrm{d}x = \int_0^{\pi/n} \sin(nx)\,\mathrm{d}x = -\frac{\cos(nx)}{n}\Big|_0^{\pi/n} = \frac{2}{n}.$$

Consider an interval $(a, b) \subseteq (0, 1)$. Say there are $k$ complete arcs of $|\sin(nx)|$ over $(a, b)$. There are an additional at most 2 fractional arcs terminating at $a$ and $b$. Since each arc contributes $\frac{2}{n}$ to the integral by the above, then

$$k\frac{2}{n} \leqslant \int_a^b |\sin(nx)|\,\mathrm{d}x \leqslant (k+2)\frac{2}{n}.$$

Each arc has length $\pi/n$, so there are at most $\frac{n}{\pi}(b-a)$ complete arcs of $|\sin(nx)|$ over $(a, b)$. Since $k$ is an integer, there are at least $\frac{n}{\pi}(b-a) - 1$ complete arcs of $|\sin(nx)|$. Therefore,

$$\left(\frac{n}{\pi}(b-a) - 1\right)\left(\frac{2}{n}\right) \leqslant k\left(\frac{2}{n}\right) \leqslant \int_a^b |\sin(nx)|\,\mathrm{d}x \leqslant (k+2)\left(\frac{2}{n}\right) \leqslant \left(\frac{n}{\pi}(b-a) + 2\right)\left(\frac{2}{n}\right).$$

Thus,

$$\frac{2(b-a)}{\pi} - \frac{2}{n} \leqslant \int_a^b |\sin(nx)|\,\mathrm{d}x \leqslant \frac{2(b-a)}{\pi} + \frac{4}{n}.$$

Taking $n \to \infty$ gives $\frac{b-a}{\pi} = \int_a^b |\sin(nx)|\,\mathrm{d}x$.

Observe now that if $I = (a, b)$ is an interval, then

$$\lim \int_0^1 \mathbb{1}_I |\sin(nx)|\,\mathrm{d}x = \frac{2}{\pi} m(I) = \frac{2}{\pi}\int \mathbb{1}_I\,\mathrm{d}x.$$

So, for any step function $\phi$, we obtain by linearity $\lim \int_0^1 \phi|\sin(nx)|\,\mathrm{d}x = \frac{2}{\pi}\int_0^1 \phi\,\mathrm{d}x$. Take $f \in L^1([0,1])$. So, $f$ may be arbitrarily approximated by step functions. Choose $\phi$ within $\epsilon$ of $f$. Then,

$$\begin{aligned}
\left|\int_0^1 f|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x\right| &= \left|\int_0^1 (f-\phi)|\sin(nx)|\,\mathrm{d}x + \int_0^1 \phi|\sin(nx)| - \frac{2}{\pi} f\,\mathrm{d}x\right| \\
&\leqslant \epsilon + \left|\int_0^1 \phi|\sin(nx)|\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x\right| \\
&= \epsilon + \left|\int_0^1 \phi|\sin(nx)| - \frac{2}{\pi}\phi\,\mathrm{d}x + \frac{2}{\pi}\int_0^1 \phi\,\mathrm{d}x - \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x\right| \\
&\leqslant \epsilon + \left|\int_0^1 \phi|\sin(nx)| - \frac{2}{\pi}\int_0^1 \phi\,\mathrm{d}x\right| + \epsilon.
\end{aligned}$$

This holds for all $\epsilon$. Taking $n \to \infty$ gives us $\int_0^1 f|\sin(nx)|\,\mathrm{d}x \to \frac{2}{\pi}\int_0^1 f\,\mathrm{d}x$. □

**Problem 5.** (Classic) Let $f \geqslant 0$ be a Lebesgue Measurable function on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f\,\mathrm{d}m = \int_0^\infty m(\{x : f(x) > t\})\,\mathrm{d}t.$$

**This solution is valid, but not as good as the one for an almost identical Problem 4 in the Analysis, Spring 2019 qual**

*Proof.* Define the function $g(t) = m(\{x : f(x) > t\})$. The claim is that $\int_{\mathbb{R}} f \, \mathrm{d}m = \int_0^\infty g \, \mathrm{d}m$. Now, first suppose that $f$ is a simple function. Write $\sum_{j=1}^n a_j \mathbb{1}_{A_j}$ in standard form. We may index such that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. Then, for $t \in (a_k, a_{k+1})$, $g(t) = \sum_{j=k+1}^n m(A_j)$. Observe that

$$\int_{a_k}^{a_{k+1}} g(t) \, \mathrm{d}t = \int_{a_k}^{a_{k+1}} \sum_{j=k+1}^n m(A_j) \, \mathrm{d}t = (a_{k+1} - a_k) \sum_{j=k+1}^n m(A_j).$$

Therefore, identifying $a_0$ with 0, we have

$$\begin{aligned} \int_0^\infty g(t) \, \mathrm{d}t &= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} g(t) \, \mathrm{d}t \\ &= \sum_{k=0}^{n-1} \left( (a_{k+1} - a_k) \sum_{j=k+1}^n m(A_j) \right) \\ &= \sum_{k=0}^{n-1} \sum_{j=k+1}^n (a_{k+1} - a_k) m(A_j) \\ &= \sum_{j=1}^n \sum_{k=0}^{j-1} (a_{k+1} - a_k) m(A_j). \end{aligned}$$

The series $\sum_{k=0}^{j-1} a_{k+1} - a_k$ is telescoping, and thus equal to $a_j - a_0 = a_j$. Therefore,

$$\int_0^\infty g(t) \, \mathrm{d}t = \sum_{j=1}^n \sum_{k=0}^{j-1} (a_{k+1} - a_k) m(A_j) = \sum_{j=1}^n a_j m(A_j) = \int f \, \mathrm{d}m,$$

given that we assume $f$ is a simple function.

Supose now that $f$ is an arbitrary Lebesgue measurable nonnegative function. Take $(\phi_n)$ to be simple functions converging monotonically to $f$. Let $g_n$ be the corresponding function $t \longmapsto m(\{x : \phi_n(x) > t\})$. The $g_n$ are also nonnegative functions. Moreover, $g_n \leqslant g_{n+1}$, for if $x$ is such that $\phi_n(x) > t$, then $\phi_{n+1}(x) > t$ as well, and so $\{x : \phi_n(x) > t\} \subseteq \{x : \phi_{n+1}(x) > t\}$. So, then $g_n$ are nonnegative and monotically increasing. Finally, we claim that $g_n \to g$. Observe that $\bigcup_{n=1}^\infty \{x \in \mathbb{R} : \phi_n(x) > t\} = \{x \in \mathbb{R} : f(x) > t\}$, for if $f(x) > t$, then there is some $n$ such that $\phi_n(x) > t$. On the other hand, since $\phi_n(x) \leqslant f(x)$ for all $n$, then if $\phi_n(x) > t$, we have $f(x) > t$. Note that by monotonicity of the $\phi_n$ we have $\bigcup_{n=1}^k \{x \in \mathbb{R} : \phi_n(x) > t\} = \{x \in \mathbb{R} : \phi_k(x) > t\}$. Therefore,

$$m(\{x \in \mathbb{R} : f(x) > t\} = m\left( \bigcup_{n=1}^\infty \{x \in \mathbb{R} : \phi_n(x) > t\} \right) = \lim_{n \to \infty} m(\{x \in \mathbb{R} : \phi_n(x) > t\}.$$

Therefore, $g(t) = \lim g_n(t)$. Thus, by MCT, we have

$$\int f \, \mathrm{d}m = \lim \int \phi_n \, \mathrm{d}m = \lim \int g_n \, \mathrm{d}m = \int g \, \mathrm{d}m,$$

where the equality $\int \phi_n \, \mathrm{d}m = \int g_n \, \mathrm{d}m$, holds since each $\phi_n$ is simple. $\square$

**Problem 6.** **(Classic)** Compute the following limit and justify your calculations

$$\lim_{n\to\infty} \int_1^n \frac{1}{\left(1+\frac{x}{n}\right)^n \sqrt[n]{x}}\,\mathrm{d}x.$$

*Proof.* We apply the binomial theorem and observe that since $x$ is nonnegative the following inequality holds for all $n \geqslant 2$:

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x}{n}\right)^k \geqslant \binom{n}{2}\left(\frac{x}{n}\right)^2 = \left(\frac{n-1}{n}\right)\frac{x^2}{2} \geqslant \frac{x^2}{4}.$$

Therefore, $4x^{-2} \geqslant (1+x/n)^{-n} \geqslant (1+x/n)^{-n}(x)^{-1/n}$ for all $x \geqslant 1$ and $n \geqslant 2$. Observe moreover that $(1+x/n)^n \to e^x$ pointwise, and $\sqrt[n]{x} \to 1$ pointwise. Therefore,

$$\mathbb{1}_{[1,n]}\frac{1}{\left(1+\frac{x}{n}\right)^n \sqrt[n]{x}} \to e^{-x}$$

pointwise. Finally, for all $x \geqslant 1$, we have

$$\int_1^\infty 4x^{-2}\,\mathrm{d}x = 4\int_1^\infty x^{-2}\,\mathrm{d}x = 4\left(-x^{-1}\Big|_1^\infty\right) = 4.$$

Thus, the improper Riemann Integral of $4x^{-2}$ is absolutely convergent, so its Lebesgue integral is finite, and thus the sequence of functions $\mathbb{1}_{[1,n]}\left(1+\frac{x}{n}\right)^{-n}(x)^{-1/n}$ has an integrable dominant for $n \geqslant 2$. So, by DCT, we obtain

$$\lim_{n\to\infty}\int_1^n \frac{1}{\left(1+\frac{x}{n}\right)^n \sqrt[n]{x}}\,\mathrm{d}x = \lim_{n\to\infty}\int_{(1,\infty)} \mathbb{1}_{[1,n]}\frac{1}{\left(1+\frac{x}{n}\right)^n}\,\mathrm{d}m = \int_1^\infty e^{-x}\,\mathrm{d}x = \frac{1}{e},$$

completing the computation. $\square$