Real Analysis Qual, Fall 2019

Problem 1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Prove that if $\lim a_n = 0$ then

$$\lim \frac{a_1 + \ldots + a_n}{n} = 0.$$

(b) Prove that if $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, then $\lim \frac{a_1+\ldots+a_n}{n} = 0$.

Proof. We prove (a). Take $\epsilon > 0$. Pick N_1 large enough such that $|a_n| < \epsilon/3$ for all $n \geqslant N_1$. Then, pick N_2 large enough such that for all $n, m \geqslant N_2$, $|a_n - a_m| < \epsilon/3$. Set $N = \max\{N_1, N_2\}$. Then, for all $n \geqslant N$, we have $|a_n| = |a_n - a_N| + |a_N| < \epsilon/3 < \epsilon$. So, for any $n \geqslant N$, observe that

$$\sum_{k=N}^{n} \left|\frac{a_k}{n}\right| < \sum_{k=N}^{n} \frac{\epsilon}{n} = \frac{n-N}{n}\epsilon \leqslant \epsilon.$$

Therefore, $\lim_{n\to\infty} \left|\sum_{k=N}^{n} \frac{a_k}{n}\right| \leqslant \epsilon$. Moreover, $\lim_{n\to\infty} \sum_{k=1}^{N} \frac{a_k}{n} = 0$. Thus,

$$\lim_{n\to\infty} \left|\sum_{k=1}^{n} \frac{a_k}{n}\right| \leqslant \lim_{n\to\infty} \sum_{k=1}^{N} \left|\frac{a_k}{n}\right| + \lim_{n\to\infty} \sum_{k=N}^{n} \left|\frac{a_k}{n}\right| \leqslant 0 + \epsilon = \epsilon.$$

Since ϵ is arbitrary, we have $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{n} = 0$.

We now prove (b). **This is giving me real trouble** □

Problem 2. Prove that $\left|\frac{d^n}{dx^n}\frac{\sin(x)}{x}\right| \leqslant \frac{1}{n}$ for all $x \neq 0$ and positive integers n.

Hint: Consider $\int_0^1 \cos(tx)\,dt$.

Proof. For $x \neq 0$, we have $\frac{\sin(x)}{x} = \int_0^1 \cos(tx)\,dt$. We show $\frac{d^n}{dx^n}\cos(tx) = (-1)^n t^n \cos(tx + \frac{n\pi}{2})$. For our base case,

$$\frac{d}{dx}\cos(tx) = -t\sin(tx) = -t\cos\left(tx + \frac{\pi}{2}\right),$$

as neeeded. Applying our inductive hypothesis,

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} &= \frac{d}{dx}\left(\frac{d^n}{dx^n}\cos(tx)\right) \\
&= \frac{d}{dx}(-1)^n t^n \cos\left(tx + \frac{n\pi}{2}\right) \\
&= (-1)^{n+1} t^{n+1} \sin\left(tx + \frac{n\pi}{2}\right) \\
&= (-1)^{n+1} t^{n+1} \cos\left(tx + \frac{(n+1)\pi}{2}\right).
\end{aligned}$$

So, the claim is proven.

Moreover, observe that $\cos(tx)$ is smooth in x on $[0, 1]$. Thus, every nth derivative $\frac{d^n}{dx^n}[\cos(tx)]$ is a smooth function over the compact domain $[0, 1]$. In particular, $\frac{d^n}{dx^n}[\cos(tx)]$

is bounded over $[0, 1]$, and thus integrable. So, we may differentiate under the integral sign to obtain

$$\frac{d}{dx}\int_0^1 \frac{d^n}{dx^n}\cos(tx)\,\mathrm{d}t = \int_0^1 \frac{d^{n+1}}{dx^{n+1}}\cos(tx)\,\mathrm{d}t.$$

Since this holds for all positive n, then in particular

$$\begin{aligned}\frac{d^n}{dx^n}\int_0^1 \cos(tx)\,\mathrm{d}t &= \int_0^1 \frac{d^n}{dx^n}\cos(tx)\,\mathrm{d}t\\ &= \int_0^1 (-1)^n t^n \cos\left(tx + \frac{n\pi}{2}\right)\mathrm{d}t\\ &\leqslant \int_0^1 \left|(-1)^n t^n \cos\left(tx + \frac{n\pi}{2}\right)\right|\mathrm{d}t\\ &\leqslant \int_0^1 t^n\,\mathrm{d}t\\ &= \frac{t^{n+1}}{n+1}\bigg|_0^1 = \frac{1}{n+1}.\end{aligned}$$

Thus, for $x \neq 0$, we obtain

$$\frac{d^n}{dx^n}\frac{\sin(x)}{x} = \frac{d^n}{dx^n}\int_0^1 \cos(tx)\,\mathrm{d}x \leqslant \frac{1}{n+1},$$

thereby proving the claim. □

Borel-Cantelli proofs are very common, and you should know them

Problem 3. Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) = 1$, $\{B_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{B}$-measurable subsets of X, and $B = \{x \in X : x \in B_n \text{ for infinitely many } n\}$.

(a) Prove that B is also a $\mathcal{B}$-measurable subset of X.

(b) Prove that if $\sum_{n=1}^{\infty}\mu(B_n) < \infty$, then $\mu(B) = 0$.

(c) Prove that if $\sum_{n=1}^{\infty}\mu(B_n) = \infty$, and the sequence of set complements $\{B_n^c\}_{n=1}^{\infty}$ satisfies

$$\mu\left(\bigcap_{n=k}^{K} B_n^c\right) = \prod_{n=k}^{K}(1 - \mu(B_n))$$

for all positive integers k and K with $k < K$, then $\mu(B) = 1$.

Hint: Use the fact that $1 - x \leqslant e^{-x}$ for all x.

Proof. If B contains x if and only if x is in infinitely many B_n, then $B = \limsup B_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_n$. So, B is measurable. Suppose that $\sum_{n=1}^{\infty}\mu(B_n) < \infty$. Observe that $\bigcup_{n=k}^{\infty} B_n \supseteq \bigcup_{n=k+1}^{\infty} B_n$. Moreover, X is finite, so $\bigcup_{n=1}^{\infty} B_n$ has finite measure. Thus, by continuity from below,

$$\mu(B) = \mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_n\right) = \lim_{n\to\infty}\mu\left(\bigcup_{k=n}^{\infty} B_n\right) = \lim_{n\to\infty}\sum_{k=n}^{\infty}\mu(B_k) = 0,$$

given that $\sum_{n=1}^{\infty} \mu(B_n)$ is convergent and thus Cauchy. So, $\mu(B) = 0$.

Finally, we have

$$\mu(B^c) = \mu\left(\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} B_n\right)^c\right) = \mu\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} \mu(B_n)^c\right) \leqslant \mu\left(\bigcup_{n=1}^{\infty} B_n^c\right) = \lim_{K\to\infty} \mu\left(\bigcup_{n=1}^{K} B_n^c\right).$$

Moreover,

$$\mu\left(\bigcup_{n=1}^{K} B_n^c\right) = \prod_{n=1}^{K}(1-\mu(B_n)) \leqslant \prod_{n=1}^{K}\exp(-\mu(B_n)) = \exp\left(-\sum_{n=1}^{K}\mu(B_n)\right).$$

Therefore,

$$\mu(B^c) = \lim_{K\to\infty} \mu\left(\bigcup_{n=1}^{K} B_n^c\right) = \lim_{K\to\infty} \exp\left(-\sum_{n=1}^{K}\mu(B_n)\right).$$

Given that $\sum_{n=1}^{\infty} \mu(B_n) = \infty$, then $-\sum_{n=1}^{\infty}\mu(B_n) = -\infty$, and so

$$\lim_{K\to\infty} \exp\left(-\sum_{n=1}^{K}\mu(B_n)\right) = 0.$$

Therefore, $\mu(B^c) = 0$, forcing $\mu(B) = \mu(X\backslash B^c) = \mu(X) - \mu(B^c) = 1$. □

Problem 4. Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a compact Hilbert space $\mathcal{H}$.

(a) Prove that for every $x \in \mathcal{H}$ one has $\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leqslant ||x||^2$.

(b) Prove that for any sequence $(a_n)_{n=1}^{\infty}$ in $\ell^2(\mathbb{N})$ there exists an element $x \in \mathcal{H}$ such that $a_n = \langle x, u_n\rangle$ for all $n \in \mathbb{N}$ and $||x||^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2$.

Proof. We prove (a). Set $y = x - \sum_{k=1}^{N}\langle x, u_k\rangle u_k$. Then, for all $1 \leqslant k \leqslant N$,

$$\langle y, u_k\rangle = \langle x - \langle x, u_k\rangle u_k, u_k\rangle = 0.$$

Therefore,

$$\|x\|^2 = \|y + \sum_{k=1}^{N}\langle x, u_k\rangle u_k\|^2 = ||y||^2 + \sum_{k=1}^{n} ||\langle x, u_k\rangle u_k||^2 = ||y^2|| + \sum_{k=1}^{n} |\langle x, u_k\rangle|^2.$$

So, we have for all N, $\sum_{k=1}^{n} |\langle x, u_k\rangle|^2 \leqslant ||x||^2$. Taking $N \to \infty$ gives the result.

We now prove (b). Set $x = \sum_{k=1}^{\infty} a_k u_k$. We first prove that $x \in \mathcal{H}$. So, consider the partial sums $s_N = \sum_{k=1}^{N} a_k u_k$. Note that for n, m with $n \geqslant m$ we have

$$||s_n - s_m||^2 = \|\sum_{k=m}^{n} a_k u_k\|^2 = \sum_{k=m}^{n} |a_k|^2 \to_{n,m\to\infty} 0,$$

given that the sequence $(a_n) \in \ell^2(\mathbb{N})$. In particular, the partial sums s_N are Cauchy, and since $\mathcal{H}$ is Hilbert, they converge to a limit. This limit is x, given that

$$\|x - s_N\|^2 = \| \sum_{k=N}^{\infty} a_k u_k\|^2 = \sum_{k=N}^{\infty} |a_n|^2 \to_{N\to\infty} 0.$$

Since, again, $(a_n) \in \ell^2(\mathbb{N})$. Thus $x \in \mathcal{H}$. Moreover,

$$\langle x, u_k\rangle = \lim\langle s_N, u_k\rangle = \lim\langle a_k u_k, u_k\rangle = a_k.$$

Finally,

$$||x||^2 = \lim ||s_N||^2 = \lim \sum_{k=1}^{N} |\langle a_k, u_k\rangle|^2 = \lim \sum_{k=1}^{N} |a_k|^2 = \lim \sum_{k=1}^{N} |\langle x, u_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2,$$

completing the proof. □

Problem 5. **(Classic)**

(a) Show that if f is continuous with compact support on $\mathbb{R}$, then

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|\, \mathrm{d}x = 0.$$

(b) Let $f \in L^1(\mathbb{R})$ and for each $h > 0$ let $\mathcal{A}_h f(x) = \frac{1}{2h} \int_{|y|\leqslant h} f(x-y)\, \mathrm{d}y$.

(i) Prove that $||\mathcal{A}_h f||_1 \leqslant ||f||_1$ for all $h > 0$.

(ii) Prove that $\mathcal{A}_h f \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

Proof. We prove (a). Let $E = \operatorname{supp} f$. Since E is compact and $|f|$ is continuous on E, then $|f|$ attains a maximum value, M. Therefore, $|f(x)| \leqslant M\mathbb{1}_E(x)$ for all x. Now, E is a compact subset of $\mathbb{R}$, so by Heine-Borel, E is bounded by a finite interval. In particular, $m(E) < \infty$, so $M\mathbb{1}_E$ is an integrable function. Observe that for all y, we have $|f(x-y) - f(x)| \leqslant 2M\mathbb{1}_E$. Moreover, $|f(x-y) + f(x)| \to_{y\to 0} 0$ pointwise. Therefore, by DCT, we obtain

$$\lim_{y\to 0} \int_{\mathbb{R}} |f(x-y) - f(x)|\, \mathrm{d}x = \int 0\, \mathrm{d}x = 0.$$

When proving b(ii), we will need this to hold more generally for $L^1(\mathbb{R})$ functions, so we prove that it does so now. Take $f \in L^1(\mathbb{R})$. Compactly supported continuous functions are dense in $L^1(\mathbb{R})$, so choose g supported continuous function. We have

$$\begin{aligned}
\int |f(x-y) - f(x)|\, \mathrm{d}x &\leqslant \int |f(x-y) - g(x-y)|\, \mathrm{d}x + \int |g(x-y) - f(x)|\, \mathrm{d}x \\
&= \int |f(x) - g(x)|\, \mathrm{d}x + \int |g(x-y) - g(x) + g(x) - f(x)|\, \mathrm{d}x \\
&= ||f - g||_1 + \int |g(x-y) - g(x)|\, \mathrm{d}x + \int |g(x) - f(x)|\, \mathrm{d}x
\end{aligned}$$

$$= 2||f-g||_1 + \int |g(x-y) - g(x)|\,\mathrm{d}x$$

So, for arbitary $\epsilon > 0$, we choose g sufficiently close to f so that $||f-g||_1 < \epsilon/4$, and then choose δ such that for all $|y| < \delta$, we have $\int |g(x-y) - g(x)|\,\mathrm{d}x < \epsilon/2$. Then, for all $|y| < \delta$, we attain

$$\int |f(x-y) - f(x)|\,\mathrm{d}x \leqslant 2||f-g||_1 + \int |g(x-y) - g(x)|\,\mathrm{d}x < \epsilon.$$

Therefore, $\int |f(x-y) - f(x)|\,\mathrm{d}x \to_{|y|\to 0} 0$, as neeeded.

Now, we prove (b). First, for part (i), we have

$$\begin{aligned}\int |\mathcal{A}_h f(x)|\,\mathrm{d}x &= \frac{1}{2h}\int \left|\int_{y\leqslant|h|} f(x-y)\,\mathrm{d}y\right|\mathrm{d}x \\ &\leqslant \frac{1}{2h}\int\int_{y\leqslant|h|} |f(x-y)|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Since $|f(x-y)|$ is a nonnegative measurable function, we apply Tonelli's Theorem to obtain

$$\frac{1}{2h}\int\int_{y\leqslant|h|} |f(x-y)|\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2h}\int_{y\leqslant|h|}\int |f(x-y)|\,\mathrm{d}x.$$

By invariance of the Lebesgue integral under shifts, we have

$$\begin{aligned}\frac{1}{2h}\int_{y\leqslant|h|}\int |f(x-y)|\,\mathrm{d}x &= \frac{1}{2h}\int_{y\leqslant|h|}\int |f(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\left(\int_{-h}^{h} 1\,\mathrm{d}y\right)\left(\int |f(x)|\,\mathrm{d}x\right) \\ &= \int |f(x)|\,\mathrm{d}x.\end{aligned}$$

Thus, $||\mathcal{A}_h(f)||_1 \leqslant ||f||_1$.

We move on to part (ii). Then,

$$\begin{aligned}\int |f(x) - \mathcal{A}_h f(x)|\,\mathrm{d}x &= \int \left|f(x) - \frac{1}{2h}\int_{|y|\leqslant h} f(x-y)\,\mathrm{d}y\right|\mathrm{d}x \\ &= \int \left|\frac{1}{2h}\int_{|y|\leqslant h} f(x)\,\mathrm{d}y - \frac{1}{2h}\int_{|y|\leqslant h} f(x-y)\,\mathrm{d}y\right|\mathrm{d}x \\ &= \int \left|\frac{1}{2h}\int_{|y|\leqslant h} f(x) - f(x-y)\,\mathrm{d}y\right|\mathrm{d}x \\ &\leqslant \frac{1}{2h}\int\int_{|y|\leqslant h} |f(x) - f(x-y)|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

The integrand is a nonnegative measurable function, so by Tonelli's Theorem, we have

$$\frac{1}{2h}\int\int_{|y|\leqslant h} |f(x) - f(x-y)|\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2h}\int_{|y|\leqslant h}\int |f(x) - f(x-y)|\,\mathrm{d}x\,\mathrm{d}y.$$

Let $\epsilon > 0$ be arbitrary. By our proof in (a), take δ small enough that for all $|y| < \delta$, we have

$$\int |f(x) - f(x-y)| \, \mathrm{d}x < \epsilon.$$

Then, set $h = \delta$. Therefore,

$$\begin{aligned} \frac{1}{2\delta} \int_{|y| \leqslant \delta} \int |f(x) - f(x-y)| \, \mathrm{d}x \, \mathrm{d}y &\leqslant \frac{1}{2\delta} \int_{|y| \leqslant \delta} \epsilon \, \mathrm{d}y \\ &= \epsilon. \end{aligned}$$

Therefore, $\mathcal{A}_h f \to f$ as $h \to 0^+$ for $f \in L^1(\mathbb{R})$. $\square$