Real Analysis Qual, Fall 2023

Problem 1. **(Classic Technique)** Let $f_n(x) = \dfrac{nx^2}{n^3 + x^3}$.

(a) Prove that f_n converge uniformly to 0 on $[0, M]$ for any $M > 0$, but does not converge uniformly to 0 on $[0, \infty)$.

(b) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ defines a continuous function on $[0, \infty)$.

Proof. We first prove (a). Over $[0, M]$, we have

$$0 \leqslant \frac{nx^2}{n^3 + x^3} \leqslant \frac{nx^2}{n^3} \leqslant \frac{x^2}{n^2} \leqslant \frac{M^2}{n^2} \to_{n\to\infty} 0.$$

Hence, $f_n \to 0$ uniformly on $[0, M]$. On the other hand, set $x_n = n$. Then,

$$\frac{nx_n^2}{n^3 + x_n^3} = \frac{n^3}{2n^3} = \frac{1}{2}.$$

Then, on $(0, \infty)$, for every n we have $f_n(x_n) = f_n(n) = \frac{1}{2}$. Therefore, on $(0, \infty)$, we cannot have $f_n \to 0$ uniformly, since for $\epsilon < 1/2$, there is no N such that for all $n \geqslant N$, we have $|f_n(x)| < \epsilon$ for all x.

For (b), first write $g(x, n) = f_n(x)$. Set $F = \sum_{n=1}^{\infty} f_n$, and observe that $F(x) = \int g(n, x)\,\mathrm{d}n$, taken with respect to the counting measure. Take $x_0 \in (0, \infty)$, and let (x_k) be a sequence converging to x_0. Then, fixing n, we have $\lim_{k\to\infty} g(n, x_k) = g(n, x_0)$, given that the $g(n, x)$ is continuous in x. Therefore, viewing $g(n, x_k)$ as a sequence of functions in k, we have $g(n, x_k) \to_{k\to\infty} g(n, x_0)$ pointwise. Furthermore, since (x_k) is convergent, it is bounded, so we may write $(x_k) \subseteq [0, M]$ for some M. By a computation above, given that $n^2 \geqslant 1$, we conclude that $g(n, x_k) \leqslant M^2$ for all n, x_k. Since M^2 is an integrable function on $[0, M]$, then by DCT we obtain

$$\lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} \int g(n, x_k)\,\mathrm{d}n = \int g(n, x_0)\,\mathrm{d}n = F(x_0).$$

Therefore, F is continuous. □

Problem 2. Let $(X, \mathcal{A})$ be a measure space, and let μ be a nonnegative set function on $\mathcal{A}$ that is finitely additive with $\mu(\emptyset) = 0$. Recall that a set function is said to be continuous from below if

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim \mu(A_j) \text{ whenever } A_j \text{ is an increasing sequence of sets in } \mathcal{A}.$$

Prove that

$$\mu \text{ is a measure} \iff \mu \text{ is continuous from below.}$$

Proof. First, say that μ is continuous from below. Then, let $E = \bigcup_{n=1}^{\infty} F_n$ for $(F_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ a countable pairwise disjoint collection of sets. Then, by continuity from below and finite additivity of μ, we have

$$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{n} F_m\right) = \lim_{n\to\infty} \mu\left(\bigcup_{m=1}^{n} F_n\right) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(F_n) = \sum_{n=1}^{\infty} \mu(F_n).$$

Therefore, μ is countably additive, and hence a measure.

Now, suppose that μ is a measure. We must show that μ is continuous from below. Say that $(F_n)_{n=1}^{\infty}$ is an increasing sequence of sets in $\mathcal{A}$. Write $E = \bigcup_{n=1}^{\infty} F_n$. Also, set $E_n = F_n \backslash F_{n-1}$, except that $E_1 = F_1$. Note that the E_n are disjoint sets contained in E. Therefore, for all m we have

$$\sum_{n=1}^{m} \mu(E_n) \leqslant \mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

On the other hand, $\bigcup_{n=1}^{m} E_n = F_1 \cup \bigcup_{n=1}^{m} (F_n \backslash F_{n-1}) = F_m$. So,

$$\mu(F_m) = \sum_{n=1}^{m} \mu(E_n) \leqslant \mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

Taking $m \to \infty$ therefore forces the equality $\lim \mu(F_m) = \mu(E)$. So, μ is continuous from below. $\square$

Problem 3. Prove that

$$1 - \frac{x^2}{2} \leqslant \cos(x) \leqslant e^{-x^2/2}$$

for all $|x| \leqslant 1$, and conclude from this that

$$\lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \int_{|x|\leqslant 1} (\cos x)^n \, \mathrm{d}x = 1.$$

Hint: You may use without proof that $\int e^{-\pi x^2} \, \mathrm{d}x = 1$.

Proof. Recall that $\cos(x)$ is analytic with taylor series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Moreover,

$$e^{(-x^2/2)} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}.$$

For any alternating series $S = \sum_{n=0}^{\infty} (-1)^n a_n$ for $a_n \geqslant 0$ such that the a_n decrease monotonically, we have $\sum_{n=0}^{2k-1} (-1)^n a_n \leqslant S \leqslant \sum_{n=0}^{2k} (-1)^n a_n$. For $x \in [-1, 1]$, the terms $x^{2n}/(2n)!$ and $x^2 n/(2^n (2n)!)$ do decrease monotonically. Hence,

$$1 - \frac{x^2}{2} \leqslant \cos(x) \leqslant 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad \text{and} \quad 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{120} \leqslant e^{-x^2/2}.$$

With these bounds,

$$e^{-x^2/2} - \cos(x) \geqslant 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{120} - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{15x^4 - x^6}{120}.$$

On $x \in (-1,1)$, $x^4 \geqslant x^4$, so $e^{-x^2/2} - \cos(x) \geqslant 0$. Therefore, we obtain $1 - x^2/2 \leqslant \cos(x) \leqslant e^{-x^2/2}$ as needed.

Now,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} (\cos(x))^n \, \mathrm{d}x \leqslant \sqrt{\frac{n}{2\pi}} \int_{-1}^{1} (e^{-x^2/2})^n \, \mathrm{d}x = \sqrt{\frac{n}{2\pi}} \int_{-1}^{1} e^{-((x\sqrt{n})/\sqrt{2})^2} \, \mathrm{d}x.$$

We make the substitution $y = \sqrt{n}x/\sqrt{2}$. Then, $(\sqrt{2}/\sqrt{n}) \, \mathrm{d}y = \mathrm{d}x$. Moreover, define $J_n = [-\sqrt{n}/\sqrt{2}, \sqrt{n}/\sqrt{2}]$. So,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} (e^{-x^2/2})^n \, \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{n}/\sqrt{2}}^{\sqrt{n}/\sqrt{2}} e^{-y^2} \, \mathrm{d}y = \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) e^{-y^2} \, \mathrm{d}y.$$

Now, $\mathbb{1}_{J_n}(y) e^{-y^2}$ is a nonnegative and monotonically increasing sequence which converges pointwise to e^{-y^2}. Therefore,

$$\lim \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) e^{-y^2} \, \mathrm{d}y = \frac{1}{\sqrt{\pi}} \int e^{-y^2} \, \mathrm{d}y = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

On the other hand,

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} (\cos(x))^n \, \mathrm{d}x \geqslant \sqrt{\frac{n}{2\pi}} \int_{-1}^{1} \left(1 - \frac{x^2}{2}\right)^n \mathrm{d}x.$$

Again, we substitute $y = \sqrt{n}x/\sqrt{2}$ and obtain

$$\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} \left(1 - \frac{x^2}{2}\right)^n \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) \left(1 - \frac{y^2}{n}\right)^n \mathrm{d}y.$$

Recall that $\lim_{n\to\infty}(1 + x/n)^n = e^x$. Substituting $x = -y^2$, we see that $\mathbb{1}_{J_n}(y)\left(1 - \frac{y^2}{n}\right)^n$ converges pointwise to e^{-y^2}. On the other hand,

$$\left(1 - \frac{y^2}{n}\right)^n \leqslant \cos\left(\frac{y}{\sqrt{n}}\right)^n \leqslant \left(e^{-(y/\sqrt{n})^2/2}\right)^n = e^{-y^2/2}.$$

Now, $e^{-y^2/2}$ is an integrable function. Hence, the functions $\mathbb{1}_{J_n}(1 - (y^2/n))^n$ have an integrable dominant. By DCT, therefore, we obtain

$$\lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int \mathbb{1}_{J_n}(y) \left(1 - \frac{y^2}{n}\right)^n \mathrm{d}y = \frac{1}{\sqrt{\pi}} \int e^{-y^2} \, \mathrm{d}y = 1.$$

Therefore, $\lim_{n\to\infty} \sqrt{n/2\pi} \int_{-1}^{1} (\cos(x))^n \, \mathrm{d}x = 1$. $\square$

Problem 4. Let $a, b > 0$. Prove that

$$\int_{[0,1]\times[0,1]} \frac{1}{x^a + y^b} \,\mathrm{d}m_2(x,y) < \infty \iff \frac{1}{a} + \frac{1}{b} > 1$$

where m_2 denotes the Lebesgue measure on $\mathbb{R}^2$.
Hint: One possible approach would be to consider separately the regions where $x^a \leqslant y^b$ and $x^a > y^b$.

Proof. Set $A = \{(x,y) \in [0,1]^2 : x^a \geqslant y^b\}$, and set $B = \{(x,y) \in [0,1]^2 : y^b \geqslant x^a\}$. Note that

$$\int_A \frac{1}{2x^a} \,\mathrm{d}m_2 \leqslant \int_A \frac{1}{x^a + y^b} \,\mathrm{d}m_2 \leqslant \int_A \frac{1}{x^a} \,\mathrm{d}m_2.$$

Likewise,

$$\int_B \frac{1}{2y^b} \,\mathrm{d}m_2 \leqslant \int_B \frac{1}{y^b + x^a} \,\mathrm{d}m_2 \leqslant \int_B \frac{1}{y^b} \,\mathrm{d}m_2.$$

Hence, since all these functions are nonnegative on $[0,1]^2$, we conclude that

$$\int \frac{1}{x^a + y^b} \,\mathrm{d}m_2 < \infty \text{ if and only if } \int_A \frac{1}{x^a} \,\mathrm{d}m_2, \int_B \frac{1}{y^b} \,\mathrm{d}m_2 < \infty.$$

Observe that (x,y) is in A if and only if $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant x^{a/b}$. Therefore, by Tonelli's Theorem,

$$\int_A \frac{1}{x^a} \,\mathrm{d}m_2 = \int_0^1 \int_0^{x^{a/b}} \frac{1}{x^a} \,\mathrm{d}y \,\mathrm{d}x = \int_0^1 x^{\frac{a}{b}-a} \,\mathrm{d}x.$$

An analogous computation shows that

$$\int_A \frac{1}{y^b} \,\mathrm{d}m_2 = \int_0^1 \int_0^{y^{a/b}} \frac{1}{y^b} \,\mathrm{d}x \,\mathrm{d}y = \int_0^1 x^{\frac{b}{a}-b} \,\mathrm{d}y.$$

These integrals are finite if and only if $\frac{a}{b} - a > 1$ and $\frac{b}{a} - b > 1$. These inequalities hold if and only if $\frac{1}{a} + \frac{1}{b} > 1$, proving the result. $\square$

Problem 5. **(Classic Technique)** Let $f_k \to f$ a.e. on $\mathbb{R}$ with $\sup_k ||f_k||_{L^2(\mathbb{R})} < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that

$$\lim_{k\to\infty} \int f_k g \,\mathrm{d}x = \int f g \,\mathrm{d}x$$

for all $g \in L^2(\mathbb{R})$.
Hint: First consider functions g supported on sets of finite measure and use Egorov's theorem.

Proof. First, $|f_k|^2 \to |f|^2$ pointwise almost everywhere. Also, since $\sup_k ||f_k||_2 < \infty$, we have $\sup_k ||f_k||_2^2 < \infty$. Therefore, by Fatou's Lemma,

$$\int |f|^2 \,\mathrm{d}x \leqslant \liminf \int |f_k|^2 \,\mathrm{d}x \leqslant \sup_k ||f_k||_2^2 < \infty.$$

Hence, $f \in L^2(\mathbb{R})$.

Suppose first that g is a compactly supported continuous function. Set $A = \operatorname{supp} g$, and observe that $g = \mathbb{1}_A g$. By Cauchy-Schwarz,

$$\left|\int fg\,\mathrm{d}x - \int f_k g\,\mathrm{d}x\right| = \left|\int (f-f_k)\mathbb{1}_A \cdot g\,\mathrm{d}x\right| \leqslant ||\mathbb{1}_A(f-f_k)||_2 \cdot ||g||_2.$$

Therefore, we prove that $||\mathbb{1}_A(f-f_k)||_2 \to 0$. Since $A \subseteq \mathbb{R}$ is compact, then in particular it is bounded, so A has finite measure. Therefore, by Egorov's theorem, there exists $E \subseteq A$ such that $m(E) < \epsilon^2$ and $f_k \to f$ uniformly on $A \backslash E$. Choose N such that for all $k \geqslant N$ we have $|f(x) - f_k(x)| < \epsilon$ for all $x \in A\backslash E$. Then,

$$||\mathbb{1}_A(f-f_k)||_2 \leqslant ||\mathbb{1}_E(f-f_k)||_2 + ||\mathbb{1}_{A\backslash E}(f-f_k)||_2 \leqslant ||\mathbb{1}_E f||_2 + ||\mathbb{1}_E f_k||_2 + ||\epsilon \mathbb{1}_{A\backslash E}||_2.$$

Write $M = \sup_k ||f_k||_2$. Then, by Cauchy-Schwarz,

$$||\mathbb{1}_E f||_2 \leqslant \sqrt{m(E)}||f||_2 < \epsilon||f||_2 \text{ and } ||\mathbb{1}_E f_k||_2 \leqslant \sqrt{m(E)}||f_k||_2 < \epsilon M.$$

Therefore,

$$||\mathbb{1}_A(f-f_k)||_2 \leqslant \epsilon(||f||_2 + M) + \epsilon||\mathbb{1}_{A\backslash E}||_2 < \infty.$$

Taking $\epsilon \to 0$ gives the result. Therefore, when g is a compactly supported continuous function,

$$\lim_{k\to\infty} \int f_k g\,\mathrm{d}x = \int fg\,\mathrm{d}x.$$

Suppose now that g is an arbitrary $L^2(\mathbb{R})$ function. Recall that compactly supported continuous functions are dense in $L^2(\mathbb{R})$. Take h within ϵ of g in the L^2 norm. Then, we have

$$\left|\int fg\,\mathrm{d}x - \int f_k g\,\mathrm{d}x\right| \leqslant \left|\int f(g-h)\,\mathrm{d}x\right| + \left|\int fh\,\mathrm{d}x - \int f_k h\,\mathrm{d}x\right| + \left|\int f_k(h-g)\,\mathrm{d}x\right|.$$

Furthermore,

$$\left|\int f(g-h)\,\mathrm{d}x\right| \leqslant ||f||_2||g-h||_2 < \epsilon||f||_2,$$

and

$$\left|\int f_k(h-g)\,\mathrm{d}x\right| \leqslant ||f_k||_2||h-g||_2 < \epsilon M.$$

Finally, choose N such that for all $k \geqslant N$, we have

$$\left|\int fh\,\mathrm{d}x - \int f_k h\,\mathrm{d}x\right| < \epsilon.$$

Then, for all $k \geqslant N$, we obtain

$$\left|\int fg\,\mathrm{d}x - \int f_k g\,\mathrm{d}x\right| \leqslant \epsilon(||f||_2 + M + 1).$$

Taking $\epsilon \to 0$ gives

$$\lim \int f_k g\,\mathrm{d}x = \int fg\,\mathrm{d}x$$

as claimed. $\square$