Real Analysis Qual, Jan 2018

Problem 1. Define

$$E := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N} \right\}.$$

Prove that $m(E) = 0$.

Proof. Define the open interval

$$E_{p,q} := \left(\frac{p}{q} - q^{-3}, \frac{p}{q} + q^{-3} \right).$$

Observe that $E = \{x \in \mathbb{R} : x \in E_{p,q} \text{ for infinitely many } p, q \in \mathbb{N}\}$. We first claim that $x \in E$ if and only if $x \in E_{p,q}$ for infinitely many distinct q. Indeed, if $x \in E_{p,q}$ for only finitely many distinct q, then there is some fixed q such that $x \in E_{p,q}$ for infinitely many p. But for p_0 fixed and any other p sufficiently large, $E_{p_0,q} \cap E_{p,q} = \emptyset$. Thus, x must be in $E_{p,q}$ for infinitely many distinct q. Set $I_n = (0, n)$. Observe that $I_n \cap E_{p,q} \neq \emptyset$ if $p/q - q^{-3} < n$, which holds if $p < nq + q^{-2} \leqslant nq + 1$. Since $p \in \mathbb{N}$, then this holds whenever $p \leqslant nq$. Note, moreover, that $E \cap I_n \subseteq \bigcup_{q=1}^{\infty} \bigcup_{p=1}^{\infty} (E_{p,q} \cap I_n)$. In fact, since $x \in E \cap I_n$ holds if $x \in E_{p,q}$ for infinitely many distinct q, then for $k \in \mathbb{N}$ we have $E \cap I_n \subseteq \bigcup_{q=k}^{\infty} \bigcup_{p=1}^{\infty} (E_{p,q} \cap I_n)$. Therefore, we obtain

$$m(E \cap I_n) = m\left(\bigcup_{q=k}^{\infty} \bigcup_{p=1}^{\infty} (E_{p,q} \cap I_n) \right) = m\left(\bigcup_{q=k}^{\infty} \bigcup_{p=1}^{nq} E_{p,q} \right) \leqslant \sum_{q=k}^{\infty} \sum_{p=1}^{nq} m(E_{p,q}).$$

Moreover, since $m(E_{p,q}) = 2q^{-3}$,

$$\sum_{q=k}^{\infty} \sum_{p=1}^{nq} m(E_{p,q}) = \sum_{q=k}^{\infty} \sum_{p=1}^{nq} 2q^{-3} = \sum_{q=k}^{\infty} 2nq^{-2} = 2n \sum_{q=k}^{\infty} q^{-2}.$$

Now, at $k = 1$, $\sum_{q=1}^{\infty} q^{-2}$ is convergent. Therefore,

$$m(E \cap I_n) \leqslant 2n \sum_{q=k}^{\infty} q^{-2} \xrightarrow[k \to \infty]{} 0.$$

So, $m(E \cap I_n) = 0$ for each I_n. Therefore, $m(E) = m(\bigcup_{n=1}^{\infty} E \cap I_n) \leqslant \sum_{n=1}^{\infty} m(E \cap I_n) = 0$. $\square$

Problem 2. Let $f_n(x) := \frac{x}{1+x^n}$, $x \geqslant 0$.

(a) This sequence of functions converges pointwise. Find its limit. Is the convergence uniform on $[0, \infty)$? Justify your answer.

(b) Compute $\lim_{n \to \infty} \int f_n \, \mathrm{d}\mu$.

Proof. For (a), observe that at $x = 1$, $f_n(x) = 1/2$, for all n. For $x > 1$, we note that with respect to n, $1 + x^n$ is an exponential function, and thus $x/(1 + x^n) \to 0$ as $n \to \infty$. For $x \in [0, 1)$, the function x^n decays to 0, so $\frac{x}{1+x^n} \to x$. Therefore, f_n converges pointwise to

$$f(x) := \begin{cases} x, \; 0 \leqslant x < 1, \\ \frac{1}{2}, \; x = 1, \\ 0, \; x > 1. \end{cases}$$

For (b), on $[1, \infty)$, $f_n(x)$ converges pointwise a.e. to f. Moreover, for $n \geqslant 3$, we have $1/x^2 \geqslant 1/x^n \geqslant 1/(1 + x^n)$. Since $1/x^2$ is Lebesgue Integrable on $[1, \infty)$, then by DCT,

$$\lim \int_1^\infty f_n \, \mathrm{d}\mu = \int_1^\infty f \, \mathrm{d}\mu = \int_1^\infty 0 \, \mathrm{d}\mu = 0.$$

On the other hand, on $[0, 1)$, $f_n(x)$ converges pointwise to f. Moreover, $x^n \geqslant x^{n+1}$, so $1/x^n \leqslant 1/x^{n+1}$, and thus $x/(1 + x^n) \leqslant x/(1 + x^{n+1})$. In particular, the f_n are monotonic nonnegative functions converging pointwise to f. So, by MCT, we obtain

$$\lim \int_0^1 f_n \, \mathrm{d}\mu = \int_0^1 f \, \mathrm{d}\mu = \int_0^1 x \, \mathrm{d}\mu = \frac{1}{2}.$$

Therefore,

$$\frac{1}{2} = \lim \int_0^1 f_n \, \mathrm{d}\mu + \lim \int_1^\infty f_n \, \mathrm{d}\mu = \lim \int_0^\infty f_n \, \mathrm{d}\mu.$$

So, we have computed the limit. □

Problem 3. **(Classic)** Let f be a nonnegative measurable function on $[0, 1]$. Show that

$$\lim_{p \to \infty} \left(\int_{[0,1]} f(x)^p \, \mathrm{d}x \right)^{1/p} = ||f||_\infty.$$

Proof. First, suppose that f is bounded. Set $a = ||f||_\infty$. Then, by definition of essential supremum, there exists some non null set A such that for all $x \in A$, $a - \epsilon < f(x)$. Therefore,

$$(a - \epsilon)\mu(A)^{1/p} = ((a - \epsilon)^p \mu(A))^{1/p} = \left(\int ((a - \epsilon)\mathbb{1}_A)^p \, \mathrm{d}\mu \right)^{1/p} \leqslant \left(\int f^p \, \mathrm{d}\mu \right)^{1/p}.$$

Since $\left(\int f^p \, \mathrm{d}\mu \right)^{1/p} \leqslant \left(\int a^p \, \mathrm{d}\mu \right)^{1/p} = a$, then

$$(a - \epsilon)\mu(A)^{1/p} \leqslant \left(\int f^p \, \mathrm{d}\mu \right)^{1/p} \leqslant a$$

for all p. Observe that $\mu(A) \leqslant 1$, so taking $p \to \infty$ we have $\mu(A)^{1/p} \to 1$. Therefore,

$$a - \epsilon \leqslant \lim_{p \to \infty} \left(\int f^p \, \mathrm{d}\mu \right)^{1/p} \leqslant a.$$

This holds for all ϵ, so we achieve $\left(\int f^p \,\mathrm{d}\mu\right)^{1/p} = a = ||f||_\infty$.

Now, suppose that $||f||_\infty = \infty$. Then, for every $a \in \mathbb{R}$, there exists a set A such that $f(x) \geqslant a$ for all $x \in A$. Therefore,

$$a\mu(A)^{1/p} \leqslant \left(\int (a\mathbb{1}_A)^p \,\mathrm{d}\mu\right)^{1/p} \leqslant \left(\int f^p \,\mathrm{d}\mu\right)^{1/p},$$

for all p. Again, $\mu(A) \leqslant 1$, so taking $p \to \infty$ we have $\mu(A)^{1/p} \to 1$. Thus, for all $a \in \mathbb{R}$,

$$a \leqslant \lim_{p\to\infty} \left(\int f^p \,\mathrm{d}\mu\right)^{1/p}.$$

So, in particular, we must have $\lim_{p\to\infty} \left(\int f^p \,\mathrm{d}\mu\right)^{1/p} = \infty$, proving the claim. □

Problem 4. Let $f \in L^2([0,1])$ and suppose that $\int_{[0,1]} f(x)x^n \,\mathrm{d}x = 0$ for all integers $n \geqslant 0$. Show that $f = 0$ a.e..

Proof. By linearity of the inner product on $L^2([0,1])$, we have $\langle f, p(x)\rangle = 0$ for all polynomials $p(x)$. We show that $f \in L^2([0,1])$ may be approximated by polynomials. Continuous functions are dense in $L^2([0,1])$. So, for $f \in L^2([0,1])$, pick a continuous g such that $||f-g||_2 < \epsilon$. By Weierstrass's Approximation Theorem, g can be approximated to arbitrary accuracy by some polynomial $p(x)$, so that for all $x \in [0,1]$, we have $|f(x) - p(x)| < \epsilon$. So,

$$||f-p||_2 \leqslant ||f-g||_2 + ||g-p||_2 \leqslant \epsilon + \left(\int_0^1 |f(x)-p(x)|^2 \,\mathrm{d}x\right)^{1/2} \leqslant \epsilon + \left(\int_0^1 \epsilon^2 \,\mathrm{d}x\right)^{1/2} = 2\epsilon.$$

Therefore, we may take a sequence $(p_n(x))$ of polynomials such that $p_n \to f$ in the L^2 norm. Then,

$$||f||_2^2 = \langle f, f\rangle = \langle \lim p_n, f\rangle = \lim\langle p_n, f\rangle = \lim 0 = 0.$$

So, $||f||_2^2 = 0$. Therefore, f is a.e. the 0 function. □

Problem 5. (Classic) Suppose $f_n, f \in L^1$, $f_n \to f$ a.e., and $\int |f_n| \,\mathrm{d}\mu \to \int |f| \,\mathrm{d}\mu$. Show that $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$.

Proof. Now, $|f_n| + f_n \to |f| + f$ pointwise, and $|f_n| - f_n \to |f| - f$ pointwise. Moreover, these are nonnegative functions. Therefore, by Fatou's Lemma,

$$\int |f| + f \,\mathrm{d}\mu \leqslant \liminf \int |f_n| + f_n \,\mathrm{d}\mu \leqslant \liminf \int |f_n| \,\mathrm{d}\mu + \liminf \int f_n \,\mathrm{d}\mu.$$

Since $\liminf \int |f_n| \,\mathrm{d}\mu = \lim \int |f_n| \,\mathrm{d}\mu = \int |f| \,\mathrm{d}\mu$, cancellation on both sides gives

$$\int f \,\mathrm{d}\mu \leqslant \liminf \int f_n \,\mathrm{d}\mu.$$

On the other hand,

$$\int |f| - f \,\mathrm{d}\mu \leqslant \liminf \int |f_n| - f_n \,\mathrm{d}\mu \leqslant \liminf \int |f_n| \,\mathrm{d}\mu + \liminf \int -f_n \,\mathrm{d}\mu.$$

Again, we may cancel on both sides to obtain

$$-\int f \,\mathrm{d}\mu \leqslant \liminf \int -f_n \,\mathrm{d}\mu \leqslant -\limsup \int f_n \,\mathrm{d}\mu.$$

Thus, in particular, $\int f \,\mathrm{d}\mu \geqslant \limsup \int f_n \,\mathrm{d}\mu$. So, we have

$$\limsup \int f_n \,\mathrm{d}\mu \leqslant \int f \,\mathrm{d}\mu \leqslant \liminf \int f_n \,\mathrm{d}\mu.$$

Thus, $\limsup \int f_n \,\mathrm{d}\mu = \liminf \int f_n \,\mathrm{d}\mu$, so that $\lim \int f_n \,\mathrm{d}\mu$ is convergent, and $\lim \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$. □