Real Analysis Qual, Spring 2019

Problem 1. Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

(a) Prove that $C([0,1])$ is complete under the uniform norm $||f||_u \coloneqq \sup_{x\in[0,1]} |f(x)|$.

(b) Prove that $C(0,1])$ is not complete under the L^1-norm $||f_1|| = \int_0^1 |f(x)|\,\mathrm{d}x$.

Proof. We begin with (a). Say that $f_n \to f$ in the uniform norm. Take $\epsilon > 0$. Choose f_n such that $||f - f_n||_u < \epsilon/4$. Pick $\delta > 0$ small enough so that for all x, y satisfying $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \epsilon/2$. Now, take $x, y \in [0,1]$ satisfying $|x - y| < \delta$. Then,

$$\begin{aligned}|f(x) - f(y)| &\leqslant |f(x) - f(y) - (f_n(x) - f_n(y))| + |f_n(x) - f_n(y)| \\ &< |f(x) - f_n(x)| + |f(y) - f_n(y)| + \epsilon/2 \\ &\leqslant 2||f - f_n||_u + \epsilon/2 \\ &< \epsilon.\end{aligned}$$

So, f is uniformly continuous on $[0,1]$, and thus $C([0,1])$ is completed under the uniform norm.

Now, we show that $C([0,1])$ is not complete under the L^1-norm. Set $a_n = \frac{1}{2} - \frac{1}{2n}$ and $b_n = \frac{1}{2} + \frac{1}{2n}$. Observe that $a_n, b_n \to 1/2$, and that $(a_n) \subseteq [0,1/2]$, $(b_n) \subseteq [1/2,1]$. Define

$$f_n(x) \coloneqq \begin{cases} 0, \text{ if } x < a_n \\ (b_n - a_n)(x - a_n), \text{ if } a_n \leqslant x \leqslant b_n, \\ 1, \text{ if } b_n < x. \end{cases}$$

Then, each f_n is piecewise continuous. Moreover, at the points a_n, b_n we have equality of the left and right limits of the distinct component functions. Therefore, f_n is a continuous function for each n.

Moreover, $f_n \to \mathbb{1}_{[1/2,1]}$ pointwise a.e.. Indeed, if $x < 1/2$, then pick a_n so that $a_n > x$. Then, $f_n(x) = 0$. Otherwise, if $x > 1/2$, pick b_n so that $b_n < x$. We then obtain $f_n(x) = 1$. Thus, $|f_n(x) - \mathbb{1}_{[1/2,1]}| \to 0$ pointwise a.e.. Moreover, $|f_n - \mathbb{1}_{[1/2,1]}|$ is measurable function on $[0,1]$ dominated by 2. Therefore, by DCT, we obtain

$$\lim \int |f_n - \mathbb{1}_{[1/2,1]}|\,\mathrm{d}m = \int 0\,\mathrm{d}m = 0.$$

So, $f_n \to \mathbb{1}_{[1/2,1]}$ in the L^1-norm, and it follows that this sequence is Cauchy in $C([0,1])$ under the L^1-norm. However, $\mathbb{1}_{[1/2,1]}$ is discontinuous. In particular, at the point $1/2$, for every δ ball we take around $1/2$, uncountably many points are sent to 0, and uncountably many are sent to 1. So, there is no redefinition on a null set making $\mathbb{1}_{[1/2,1]}$ continuous. Therefore, $C[(0,1)]$ is not complete under the uniform norm. $\square$

Problem 2. Let $\mathcal{B}$ denote the set of all Borel subsets of $\mathbb{R}$ and $\mu : \mathcal{B} \longrightarrow [0,\infty)$ denote a finite Borel measure on $\mathbb{R}$.

(a) Prove that if $\{F_k\}$ is a sequence of Borel sets for which $F_k \supseteq F_{k+1}$ for all k, then

$$\lim_{k\to\infty} \mu(F_k) = \mu\left(\bigcap_{k=1}^{\infty} F_k\right).$$

(b) Suppose μ has the property that $\mu(E) = 0$ for every $E \in \mathcal{B}$ with Lebesgue measure $m(E) = 0$. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ so that if $E \in \mathcal{B}$ with $m(E) < \delta$, then $\mu(E) < \epsilon$.

Proof. We start with (a). Let $(E_n)_{n=1}^{\infty}$ be a sequence of Borel sets such that $E_j \subseteq E_{j+1}$. Set $F_k = E_k \backslash \bigcup_{n=1}^{k-1} E_n$. Observe that $\bigcup_{k=1}^{\infty} F_k = \bigcup_{n=1}^{\infty} E_n$. Moreover, $E_n = \bigcup_{k=1}^{n} F_k$. Since the F_k are disjoint, we in fact have $\mu(E_n) = \sum_{k=1}^{n} \mu(F_k)$. Now, again by disjointness of the F_k,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k).$$

Observe that since μ is finite, then this sum is finite. In particular, the series is Cauchy, so there exists some j large enough such that for any $\epsilon > 0$, we obtain $\sum_{k=j}^{\infty} \mu(F_k) < \epsilon$. Thus, for all $\epsilon > 0$, for j large enough, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) - \epsilon < \sum_{k=1}^{j} \mu(F_k) = \mu(E_j).$$

Since $\mu(E_j) \leqslant \mu(\bigcup_{n=1}^{\infty} E_n)$, we obtain $\lim_{j\to\infty} \mu(E_j) = \mu(\bigcup_{n=1}^{\infty} E_n)$. Now, say that $(F_k)_{k=1}^{\infty}$ satisfies $F_k \supseteq F_{k+1}$. Set $E_n = F_1 \backslash F_n$. Since $F_n \supseteq F_{n+1}$ for each n, then $E_n \subseteq E_{n+1}$. Therefore, using that μ is a finite measure, we have

$$\mu(F_1) - \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu\left(F_1 \backslash \bigcap_{k=1}^{\infty} F_k\right) = \mu\left(\bigcup_{k=1}^{\infty} (F_1 \backslash F_k)\right) = \lim \mu(E_k) = \mu(F_1) - \lim \mu(F_k).$$

After the proper manipulations we have $\lim \mu(F_k) = \mu(\bigcap_{k=1}^{\infty} F_k)$.

Now we prove (b). Suppose that the statement does not hold. So, there is an $\epsilon > 0$ such that for every δ there exists an $E \in \mathcal{B}$ such that $m(E) < \delta$ but $\mu(E) > \epsilon$. Construct a sequence of sets E_n by choosing E_n so that $m(E_n) < 2^{-n}$ but $\mu(E_n) > \epsilon$. Define $F_k = \bigcup_{n=k}^{\infty} E_n$. Observe that $F_k \supseteq F_{k+1}$. Moreover,

$$m(F_1) \leqslant \sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Therefore, m restricted to F_1 is a finite measure space. By (b), we obtain $m(\bigcap_{k=1}^{\infty} F_k) = \lim m(F_k) = 0$. On the other hand, also by (b), we have $\mu(\bigcap_{k=1}^{\infty} F_k) = \lim \mu(F_k)$. Since $\mu(F_k) \geqslant \epsilon$ for each k, then $\mu(\bigcap_{k=1}^{\infty} F_k) \geqslant \epsilon$. A contradiction, for we have found a set A such that $m(A) = 0$, but $\mu(A) \neq 0$. $\square$

Problem 3. Let (f_k) be any sequence of functions in $L^2([0,1])$ satisfying $||f_k||_2 \leqslant M$ for all $k \in \mathbb{N}$. Prove that if $f_k \to f$ almost everywhere, then $f \in L^2([0,1])$ with $||f||_2 \leqslant M$ and

$$\lim \int_0^1 f_k(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x.$$

Hint: Try using Fatou's Lemma to show that $||f_2|| \leqslant M$ and then try applying Egorov's Theorem.

Proof. Since $f_k \to f$ pointwise a.e., then f_k^2 is a sequence of nonnegative measurable functions converging pointwise a.e. to f^2. Moreover, $||f_k||_2^2 \leqslant M^2$. So, in particular, the functions f_k^2 are dominated by M^2 on $[0,1]$. Since M^2 is integrable on $[0,1]$, then by DCT we have

$$\lim \int_0^1 f_k^2\,\mathrm{d}x = \int_0^1 f^2\,\mathrm{d}x.$$

We take the square root of both sides, and observe that by continuity of the square root, we have the convergence $||f_k||_2 \to ||f||_2$, and thus $||f||_2 \leqslant M$. In particular, $f \in L^2([0,1])$.

Now, choose E so that $f_k \to f$ uniformly on E^c, and $m(E) < \epsilon$. Pick k large enough so that $|f_k - f| < \epsilon$. Then,

$$\int_0^1 |f_k - f|\,\mathrm{d}x = \int_{E^c} |f_k - f|\,\mathrm{d}x + \int_E |f_k - f|\,\mathrm{d}x < \int_{E^c} \epsilon\,\mathrm{d}x + \int_0^1 \mathbb{1}_E |f_k - f|\,\mathrm{d}x.$$

Moreover, by Cauchy-Schwarz,

$$\int_0^1 \mathbb{1}_E |f_k - f|\,\mathrm{d}x \leqslant ||\mathbb{1}_E||_2 \cdot (||f_k - f||_2) \leqslant m(E)(||f_k||_2 + ||f||_2) < 2\epsilon M.$$

Therefore,

$$\int_0^1 |f_k - f| < \int_0^1 \epsilon\,\mathrm{d}x + 2\epsilon M = \epsilon + 2\epsilon M.$$

So, $f_k \to f$ in L^1. Moreover,

$$\lim \left| \int_0^1 f_k - f\,\mathrm{d}x \right| \leqslant \lim \int_0^1 |f_k - f|\,\mathrm{d}x = 0.$$

So, $\lim \int_0^1 f_k\,\mathrm{d}x = \int_0^1 f\,\mathrm{d}x$. ☐

Note: The following problem is a common problem. It can also be found as Problem 5 in the Fall 2018 Qual.

Problem 4. Let f be a nonnegative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leqslant t \leqslant f(x)\}$. Prove the validity of the following two statements.

(a) f is a Lebesgue measurable function on $\mathbb{R}^n \iff \mathcal{A}$ is Lebesgue measurable on $\mathbb{R}^{n+1}$.

(b) If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geqslant t\})\,\mathrm{d}t.$$

Proof. We begin with (a). If $\mathcal{A}$ is measurable, then each t-section $\mathcal{A}^t$ is measurable. Now, $\mathcal{A}^t = \{x \in \mathbb{R}^n : f(x) \geqslant t\} = f^{-1}([t,\infty))$. Therefore, the pre-images of $[t,\infty)$ are measurable for each $t \in [0,\infty)$. Since the intervals $[t,\infty)$ generate a sub-algebra of Lebesgue measurable sets consisting of all nonnegative sets, and $f^{-1}(B) = \emptyset$ for $B \subseteq (-\infty, 0)$, then f is measurable on this sub-algebra, and thus Lebesgue measurable. Suppose on the contrary that f is measurable. Define the set $B = \{(y,t) \in \mathbb{R} \times \mathbb{R}^{\geqslant 0} : y \geqslant t\}$. Observe that B is the pre-image of $[0,\infty)$ for the measurable function $(y,t) \longmapsto y - t$, and thus B is measurable, so $\mathbb{1}_B$ is a measurable function. Moreover, $(f(x), t)$ is a product of measurable functions and thus measurable, so $g : (x,t) \longmapsto \mathbb{1}_B(f(x),t)$ is a measurable function. Now, the pre-image $g^{-1}(\{1\})$ consists of all those points satisfying $\mathbb{1}_B(f(x),t) = 1$, which is all points satisfying $f(x) \geqslant t \geqslant 0$, and this is $\mathcal{A}$. Since $\{1\}$ is measurable, then $\mathcal{A}$ is measurable.

We now prove (b). First, note that $\mathbb{1}_A(x,t) = \mathbb{1}_{(0,f(x))}(t) = \mathbb{1}_B(f(x),t) = g(x,t)$. Therefore, applying Tonelli, we have

$$m(A) = \int_{R^{n+1}} \mathbb{1}_A \,\mathrm{d}(m_n \times m)(x,t) = \int_{\mathbb{R}^n \times \mathbb{R}} g(x,t) \,\mathrm{d}(m_n \times m)(x,t) = \int_0^\infty \int_{\mathbb{R}^n} g(x,t) \,\mathrm{d}x \,\mathrm{d}t.$$

For t fixed, $g(x,t) = \mathbb{1}_{[t,\infty)}(f(x))$. So, with $t \geqslant 0$ fixed, g is an indicator function of the set $\{x \in \mathbb{R}^n : f(x) \geqslant t\}$. Therefore,

$$\int_{\mathbb{R}^n} g \,\mathrm{d}x = m(\{x \in \mathbb{R}^n : f(x) \geqslant t\}), \text{ so } m(A) = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geqslant t\}) \,\mathrm{d}t.$$

We may again apply Tonelli's Theorem to switch the order of integration. We have,

$$m(A) = \int_{\mathbb{R}^n} \int_0^\infty g(x,t) \,\mathrm{d}t \,\mathrm{d}x.$$

Now, for x fixed, $g(x,t) = \mathbb{1}_{[0,f(x))}(t)$. That is, $g(x,t)$ is the indicator function of the interval $[0, f(x))$. So,

$$\int_0^\infty g(x,t) \,\mathrm{d}t = \int_0^\infty \mathbb{1}_{[0,f(x))}(t) \,\mathrm{d}t = m([0,f(x))) = f(x).$$

Therefore,

$$m(A) = \int_{\mathbb{R}^n} \int_0^\infty g(x,t) \,\mathrm{d}t \,\mathrm{d}x = \int_{\mathbb{R}^n} f(x) \,\mathrm{d}x.$$

This completes the proof. ☐

Problem 5.

(a) Show that $L^2([0,1]) \subseteq L^1([0,1])$ and argue that $L^2([0,1])$ in fact forms a dense subset of $L^1([0,1])$.

(b) Let Λ be a continuous linear functional on $L^1([0,1])$.

Prove the *Riesz Representation Theorem* for $L^1([0,1])$ by following the steps below:

(i) Establish the existence of a function $g \in L^2([0,1])$ which represents Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)} \,\mathrm{d}x \quad \text{for all } f \in L^2([0,1]).$$

(ii) Argue that the g obtained above must in fact belong to $L^\infty([0,1])$ and represent Λ in the sense that

$$\Lambda(f) = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x \ \text{ for all } f \in L^1([0,1])$$

with

$$||g||_{L^\infty([0,1])} = ||\Lambda||_{L^1([0,1])^*}.$$

Proof. We first prove (a). Take $f, 1 \in L^2([0,1])$. Then, $|f| \in L^2([0,1])$. Therefore, applying Cauchy-Schwarz, we have

$$||f||_1 = \int_0^1 |f| \cdot 1\,\mathrm{d}m \leqslant ||f||_2||1||_2 = ||f||_2 < \infty.$$

Thus, $f \in L^1([0,1])$. Density follows, since simple functions are dense in both $L^1([0,1])$ and $L^2([0,1])$, and thus $L^2([0,1])$ is dense in $L^1([0,1])$.

We now prove (b). So, we show that Λ is a bounded linear functional on $L^2([0,1])$. Since $L^2([0,1])$ is a subspace fo $L^1([0,1])$, then Λ is at least a linear functional on $L^2([0,1])$. Moreover, it is bounded, since for all $f \in L^2([0,1])$, we have

$$|\Lambda(f)| \leqslant ||\Lambda||_{L^1([0,1])^*}||f||_1 \leqslant ||\Lambda||_{L^1([0,1])^*}||f||_2$$

since in (a) we showed that $||f||_1 \leqslant ||f||_2$. Therefore, Λ is a bounded linear functional on $L^2([0,1])$. So, by the Reisz Representation Theorem, there exists some $g \in L^2([0,1])$ such that

$$\Lambda(f) = \int_0^1 f\overline{g}\,\mathrm{d}m$$

for all $f \in L^2([0,1])$.

Let $E = \{x \in [0,1] : g(x) \neq 0\}$. If $m(E) = 0$, then g is a.e. 0, so Λ is 0 on a dense subset of $L^1([0,1])$, and thus the 0 functional. Otherwise, define h_A so that $h|_{E^c} = 0$ and $h_A|_E = \frac{\mathbb{1}_A g}{m(A)|g|}$ for A not a null set. Since E is measurable, then h_A is measurable for measurable A. Note that $|h_A|$ is a simple function, so h_A is in $L^2([0,1])$. Moreover,

$$\int_0^1 |h_A|\,\mathrm{d}m = \int_E \frac{\mathbb{1}_A|g|}{m(A)|g|}\,\mathrm{d}m = \frac{1}{m(A)}\int_E \mathbb{1}_A\,\mathrm{d}m = \frac{1}{m(A)}\int \mathbb{1}_A\,\mathrm{d}m = 1.$$

In particular, $||h_A||_1 = 1$. Take $A \subseteq [0,1]$ to be a non null set and suppose that $g|_A \geqslant a > 0$. Then,

$$||\Lambda||_{L^1([0,1])^*} \geqslant |\Lambda(h_A)| = \left|\int_E \frac{\mathbb{1}_A g}{m(A)|g|}\overline{g}\,\mathrm{d}m\right| = \left|\int \frac{\mathbb{1}_A|g|}{m(A)}\,\mathrm{d}m\right| \geqslant \int \frac{\mathbb{1}_A a}{m(A)}\,\mathrm{d}m = a.$$

Therefore, $||g||_\infty \leqslant ||\Lambda||_{L^1([0,1])^*}$. Thus, $g \in L^\infty([0,1])$. Now we show that $||g||_\infty = ||\Lambda||_{L^1([0,1])^*}$. So, for any $f \in L^1([0,1])$, we have $h_n \to f$ with $h_n \in L^2([0,1])$. Thus,

$$|\Lambda(h_n)| = \left|\int_0^1 h_n\overline{g}\,\mathrm{d}m\right| \leqslant \int_0^1 |h_n\overline{g}|\,\mathrm{d}m \leqslant \int_0^1 |h_n|||g||_\infty\,\mathrm{d}m = ||g||_\infty \cdot ||h_n||_1.$$

By continuity, we have $|\Lambda(f)| = \lim |\Lambda(h_n)| \leqslant \lim ||g||_\infty ||h_n||_1 = ||g||_\infty ||f||_1$. That is, $||\Lambda||_{L^1([0,1])^*} \leqslant ||g||_\infty$, giving equality.

We prove the last part of (b). So, let $f \in L^1([0,1])$. By density of $L^2([0,1])$, taking $h_n \to f$ with $h_n \in L^2([0,1])$, we obtain

$$\left| \int_0^1 f\overline{g}\,\mathrm{d}m - \int_0^1 h_n\overline{g}\,\mathrm{d}m \right| = \left| \int_0^1 (f - h_n)\overline{g}\,\mathrm{d}m \right| \leqslant ||g||_\infty \int_0^1 |f - h_n|\,\mathrm{d}m.$$

Since $h_n \to f$ in $L^1([0,1])$, then we have $\Lambda(h_n) = \int_0^1 h_n\overline{g}\,\mathrm{d}m \to \int_0^1 f\overline{g}\,\mathrm{d}m$. By the continuity of Λ, we also obtain $\Lambda(h_n) \to \Lambda(f)$. Therefore,

$$\Lambda(f) = \lim \Lambda(h_n) = \int_0^1 f\overline{g}\,\mathrm{d}m,$$

completing the proof. $\square$