Real Analysis Qual, Spring 2021

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E_n \in \mathcal{M}$ be a measurable set for $n \geqslant 1$. Let $f_n = \chi_{E_n}$ be the indicator function of the set E_n. Prove that

(a) $f_n \to 1$ uniformly if and only if there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n \geqslant N$.

(b) $f_n(x) \to 1$ for almost every x if and only if

$$\mu\left(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X \backslash E_k)\right) = 0.$$

Proof. We prove (a). First suppose that $f_n \to 1$ uniformly. Then, there is some N such that for all $n \geqslant N$, we have $|1 - \chi_{E_n}(x)| < 1/2$ for all x. If there is some x such that $x \notin E_n$, then $|1 - \chi_{E_n}(x)| = |1 - 0| = 1 \not< 1/2$. Therefore, we have $E_n = X$ for all $n \geqslant N$. For the reverse direction, suppose there is some n such that $E_n = X$ for all $n \geqslant N$. Then, take $\epsilon > 0$ to be arbitrary. For all $n \geqslant N$, we have $|1 - \chi_{E_n}(x)| = |1 - \chi_X(x)| = 0 < \epsilon$. So, $f_n \to 1$ uniformly.

Now we prove (b). Suppose that $f_n(x) \to 1$ for a.e. x. Observe $x \in \bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)$ if for all $n \geqslant 0$, there exists some $k \geqslant n$, such that $x \in X\backslash E_k$. In particular, for all $n \geqslant 0$, there exists $k \geqslant 0$ such that $x \notin E_k$, so that $f_k(x) = 0$. Therefore, $f_k(x) \not\to 1$ as $k \to \infty$. Thus, x belongs to the set of points A such that $f_n(x) \not\to 1$. Since $f_n(x) \to 1$ for a.e. x, then A has measure 0, so

$$\mu\left(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X \backslash E_k)\right) \leqslant \mu(A) = 0,$$

as needed.

Suppose alternatively that $\mu(\bigcap_{n\geqslant 0}\bigcup_{k\geqslant n}(X\backslash E_k)) = 0$. Consider the set of points A such that $f_n(x) \not\to 1$ as $n \to \infty$. Suppose that $x \in A$. Then, for all $\epsilon > 0$, there is no $n \geqslant 0$, such that for all $k \geqslant n$ we have $|1 - f_n(x)| < \epsilon$. In particular, choosing $\epsilon = 1/2$, for all $n \geqslant 0$, there is some $k \geqslant n$ such that$|1 - f_k(x)| > 1/2$. Then, $f_n(x) \neq 1$, so $f_k(x) = 0$, and thus $x \in X\backslash E_k$. So, for all $n \geqslant 0$, there is some $k \geqslant n$ such that $x \in X\backslash E_k$. So, for all $n \geqslant 0$, we have $x \in \bigcup_{k\geqslant n} X\backslash E_k$, and thus $x \in \bigcap_{n\geqslant 0}\bigcup_{x\geqslant n} X\backslash E_k$. This is a 0 measure set, and it contains A, so A is a 0 measure set. □

Problem 2. **(Classic)** Calculate the limit

$$L := \lim_{n\to\infty}\int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)}\,\mathrm{d}x.$$

Proof. Pointwise $\mathbb{1}_{[0,n]}\cos(x/n) \to 1$, since $x/n \to 0$ pointwise and $\cos(0) = 1$. Therefore, for all $x > 0$ we have

$$\lim_{n\to\infty}\mathbb{1}_{[0,n]}\frac{\cos(x/n)}{x^2 + \cos(x/n)} = \frac{1}{x^2 + 1}.$$

Since $3/2 < \pi/2$, then on the interval $[0, 3/2]$, $\cos(x/n)$ decreases monotonically for all n. Moreover, since $3/2 \leqslant 3/2n$ for all n, then we have $\cos(3/2n) \geqslant \cos(3/2)$ for all n. Thus, for all n, on the interval $[0, 3/2]$,

$$\left|\frac{\cos(x/n)}{x^2 + \cos(x/n)}\right| \leqslant \frac{1}{x^2 + \cos(x/n)} \leqslant \frac{1}{x^2 + \cos(3/2)}.$$

Moreover, for all n we have $\cos(x/n) \geqslant -1$. Therefore, for $x \in [3/2, \infty)$ we have

$$\left| \frac{\cos(x/n)}{x^2 + \cos(x/n)} \right| \leqslant \frac{1}{x^2 + \cos(x/n)} \leqslant \frac{1}{x^2 - 1}.$$

Define

$$f(x) = \begin{cases} \frac{1}{x^2 + \cos(3/2)}, & \text{if } x \in [0, 3/2], \\ \frac{1}{x^2 - 1}, & \text{if } x \in (3/2, \infty). \end{cases}$$

Note that $|f| = f$, and that, by the foregoing, we have

$$\left| \frac{\cos(x/n)}{x^2 + \cos(x/n)} \right| \leqslant f(x)$$

for all x. Moreover, f is Lebesgue integrable on $[0, \infty]$. Indeed,

$$\int_0^\infty |f(x)| \, \mathrm{d}x = \int_0^{3/2} f(x) \, \mathrm{d}x + \int_{3/2}^\infty f(x) \, \mathrm{d}x = \int_0^{3/2} \frac{1}{x^2 + \cos(3/2)} \, \mathrm{d}x + \int_{3/2}^\infty \frac{1}{x^2 - 1} \, \mathrm{d}x.$$

Both these integrals are finite, so f is Lebesgue integrable as claimed. Therefore, by DCT,

$$\lim_{n \to \infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)} \, \mathrm{d}x = \int_0^\infty \frac{1}{x^2 + 1} \, \mathrm{d}x.$$

Now, observe that $\tan : [0, 2\pi) \longrightarrow [0, \infty)$ is a diffeomorphism. So, we perform the substitution $x = \tan(\theta)$. Note that $\mathrm{d}x = \sec^2(\theta) \, \mathrm{d}\theta$. Therefore, since $\tan^2(\theta) + 1 = \sec^2(\theta)$, we have

$$\lim_{n \to \infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)} \, \mathrm{d}x = \int_0^\infty \frac{1}{x^2 + 1} \, \mathrm{d}x = \int_0^{2\pi} \frac{1}{\tan^2(\theta) + 1} \sec^2(\theta) \, \mathrm{d}\theta = \int_0^{2\pi} 1 \, \mathrm{d}x = 2\pi.$$

□

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $(f_n)_{n=1}^\infty \subseteq L^1(X, \mu)$ and $f \in L^1(X, \mu)$ such that $f_n(x) \to x$ as $n \to \infty$ for almost every $x \in X$. Prove that for every $\epsilon > 0$ there exists $M > 0$, and a set $E \subseteq X$, such that $\mu(E) < \epsilon$ and $|f_n(x)| \leqslant M$ for all $x \in X \backslash E$ and $n \in \mathbb{N}$.

Proof. Define $A_m = \{x \in X : \exists n \in \mathbb{N}, |f_n(x)| > m\}$. Observe that the A_m are monotone, since if $x \in A_{m+1}$, there is some n such that $|f_n(x)| > m + 1 \geqslant m$, and so $x \in A_m$. For $x \in \bigcap_{m=1}^\infty A_m$, the sequence $(f_n(x))$ does not converge, since for every M, there exists some n such that $|f_n(x)| \geqslant M$. Since f_n converges pointwise a.e., then $\bigcap_{m=1}^\infty A_m$ must be a null set. Since X is a finite measure space, and so A_1 in particular is finite, then by continuity from below

$$0 = \mu\left(\bigcap_{m=1}^\infty A_m\right) = \lim \mu(A_m).$$

So, let $\epsilon > 0$ be arbitray. Pick M so that $\mu(A_M) < \epsilon$. Then, for $x \in X \backslash A_M$, we must have $|f(x)| \leqslant M$. Thus, $A_M = E$ satisfies the requirements of our set, proving the claim. □

Problem 4. **(Classic Technique)** Let f and g be Lebesgue Integrable on $\mathbb{R}$. Let $g_n(x) = g(x-n)$. Prove that

$$\lim_{n\to\infty} ||f - g_n||_1 = ||f||_1 + ||g||_1.$$

Proof. We first suppose that f, g are continuous functions with compact support. Since supp f, supp g compact, then they are bounded. So, say that supp f is supported on $[-N, N]$ and that supp g is supported on $[-M, M]$. Observe that if $g_n(x) \neq 0$, then $g(x-n) \neq 0$, and this happens if and only if $x - n \in [-M, M]$ which occurs if and only if $x \in [n-M, n+M]$. So, $g_n(x)$ is supported on $[n-M, n+M]$. Take $n \geqslant N+M$. Then, $[n-M, n+M] \cap [-N, N] = \emptyset$, and so $f(x) \neq 0$ if and only if $g_n(x) = 0$, and vice versa. Therefore,

$$\int |f - g_n| \,\mathrm{d}x = \int_{-N}^{N} |f - g_n| \,\mathrm{d}x + \int_{n-M}^{n+M} |f - g_n| \,\mathrm{d}x = \int_{-N}^{N} |f| \,\mathrm{d}x + \int_{n-M}^{n+M} |g_n| \,\mathrm{d}x.$$

Now, applying the change of coordinates $y = x - n$, we have

$$\int_{n-M}^{n+M} |g_n| \,\mathrm{d}x = \int \mathbb{1}_{[-M,M]}(x-n)|g(x-n)| \,\mathrm{d}x = \int \mathbb{1}_{[-M,M]}(y)|g(y)| \,\mathrm{d}y = \int_{-M}^{M} |g| \,\mathrm{d}x.$$

So,

$$\int |f - g_n| \,\mathrm{d}x = \int_{-N}^{N} |f| \,\mathrm{d}x + \int_{-M}^{M} |g| \,\mathrm{d}x = \int |f| \,\mathrm{d}x + \int |g| \,\mathrm{d}x$$

for all n sufficiently large.

Now, take f, g to be arbitrary L^1 functions. Take $\epsilon > 0$. Let ϕ be within $\epsilon/4$ of f, and let ψ be within $\epsilon/4$ of g in the L^1-norm. Then, choose n sufficiently large that $||\phi - \psi_n|| = ||\phi||_1 + ||\psi||_1$, which is possible by the above proof. Observe that using the proper change of coordinates, we have $||g_n - \psi_n|| = ||g - \psi||$. So,

$$\Big| ||f - g_n|| - ||\phi - \psi_n|| \Big| \leqslant ||f - g_n - (\phi - \psi_n)|| \leqslant ||f - \phi|| + ||g_n - \psi_n|| < \epsilon/2.$$

We have

$$\Big| ||f - g_n|| - ||\phi|| - ||\psi|| \Big| \leqslant \Big| ||f - g_n|| - ||\phi - \psi_n|| \Big| + \Big| ||\phi - \psi_n|| - ||\phi|| - ||\psi|| \Big| = \epsilon/2.$$

So, for all n sufficiently large,

$$\Big| ||f - g_n|| - ||f|| - ||g|| \Big| \leqslant \Big| ||f - g_n|| - ||\phi|| - ||\psi|| \Big| + \Big| ||\phi|| + ||\psi|| - ||f|| - ||g|| \Big| \leqslant \epsilon/2 + \epsilon/2 = \epsilon,$$

completing the proof. □

Problem 5. Let $f_n \in L^2([0,1])$ for $n \in \mathbb{N}$. Assume that

(a) $||f_n||_2 \leqslant n^{-51/100}$, for all $n \in \mathbb{N}$, and

(b) $\hat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, that is

$$\hat{f}_n(k) = \int_0^1 f(x) e^{-2\pi i k x} \,\mathrm{d}x = 0, \text{ for } k \notin [2^n, 2^{n+1}].$$

Prove that $\sum_{n=1}^{\infty} f_n$ converges in the Hilbert space $L^2([0,1])$.

Proof. We prove that (f_n) is an orthogonal sequence. Say that $m \neq n$. Then,

$$\langle f_n, f_m \rangle = \sum_{k=1}^{\infty} \langle f_n e^{-2\pi ikx}, f_m e^{-2\pi ikx} \rangle \leqslant \sum_{k=1}^{\infty} ||f_n e^{-2\pi ikx}||_2 ||f_m e^{-2\pi ikx}||_2.$$

Observe that $||f_n e^{-2\pi ikx}|| = |\hat{f}_n(k)|$. Moreover, $\hat{f}_n(k)\hat{f}_m(k) \neq 0$ if and only if $k \in [2^n, 2^{n+1}]$ and $k \in [2^m, 2^{m+1}]$. If $m \neq n$, then these two intervals are disjoint, and so

$$\langle f_n, f_m \rangle = \sum_{k=1}^{\infty} ||f_n e^{-2\pi ikx}||_2 ||f_m e^{-2\pi ikx}||_2 \sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} |\hat{f}_n(k)\hat{f}_m(k)| = \sum_{k=1}^{\infty} 0 = 0.$$

So, the f_n are orthogonal as claimed. Therefore, by the Pythagorean Theorem

$$\left\| \sum_{n=k}^{N} f_n \right\|_2^2 = \sum_{n=k}^{N} ||f_n||_2^2 \leqslant \sum_{n=k}^{N} n^{-102/100}.$$

Taking $N \to \infty$, we have

$$\left\| \sum_{n=k}^{\infty} f_n \right\|_2^2 \leqslant \sum_{n=k}^{\infty} n^{-102/100} < \infty.$$

For all k, this is a p-series, and thus convergent. Moreover, we have $\sum_{n=k}^{\infty} n^{-102/100} \to_{k\to\infty} 0$. Therefore, $S_N = \sum_{n=1}^{N} f_n$ is a Cauchy sequence. Since Hilbert spaces are complete, then the S_N converge. □

Problem 6. Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function, and for $x \in \mathbb{R}$ define the set

$$E_x \coloneqq \{y \in \mathbb{R} : m(\{x \in \mathbb{R} : f(x,z) = f(x,y)\}) > 0\}.$$

Show that

$$E \coloneqq \bigcup_{x \in \mathbb{R}} \{x\} \cup E_x$$

is a measurable subset of $\mathbb{R} \times \mathbb{R}$.
Hint: Consider the measurable function $h(x,y,z) \coloneqq f(x,y) - f(x,z)$.

There is some lore behind this problem. It is a known hard problem. It went unsolved during the qual, and I do not know if a solution is known. I have not tried to solve it, and I don't think you should owrry about this question.