Real Analysis Qual, Spring 2022

Problem 1. **(Classic)** Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$$

defines a function that is differential with continuous derivative on $(0, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} \frac{1-n^2x^2}{(1+n^2x^2)^2}$$

on $(0, \infty)$.

Proof. First, if x is fixed, then

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2} \leqslant \sum_{n=1}^{\infty} \frac{x}{n^2x^2} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

So, f is a real-valued function for all $x \in (0, \infty)$. Define

$$f(n,x) \coloneqq \frac{x}{1+n^2x^2}, \text{ and observe that for all } x \in (0,\infty), \; f(x) = \int f(n,x)\,\mathrm{d}n$$

taken with respect to the counting measure. We claim that $x \longmapsto \frac{\partial}{\partial x} f(n,x)$ is an $L^1((0,\infty))$ function for all n. So, for all $x \in (0,\infty)$,

$$\begin{aligned}\left|\frac{\partial}{\partial x} f(n,x)\right| &= \left|\frac{1-n^2x^2}{(1+n^2x^2)^2}\right| \\ &\leqslant \left|\frac{1}{(1+n^2x^2)^2}\right| + \left|\frac{n^2x^2}{(1+n^2x^2)^2}\right|.\end{aligned}$$

Since, for all x, we have $1/(1+n^2x^2) \leqslant 1$, we obtain

$$\begin{aligned}\left|\frac{1}{(1+n^2x^2)^2}\right| + \left|\frac{n^2x^2}{(1+n^2x^2)^2}\right| &\leqslant \frac{1}{1+n^2x^2} + \frac{n^2x^2}{n^4x^4+2n^2x^2+1} \\ &\leqslant \frac{1}{1+n^2x^2} + \frac{n^2x^2}{n^4x^4+2n^2x^2}.\end{aligned}$$

Finally,

$$\frac{1}{1+n^2x^2} + \frac{n^2x^2}{n^4x^4+2n^2x^2} = \frac{1}{1+n^2x^2} + \frac{1}{n^2x^2+2} \leqslant \frac{2}{1+n^2x^2} \leqslant \frac{2}{1+x^2}.$$

On $(0, 1)$, we have $\frac{2}{1+x^2} \leqslant 2$. On $[1, \infty)$, we have the absolutely convergent integral

$$\int_1^{\infty} \frac{2}{1+x^2}\,\mathrm{d}x \leqslant \int_1^{\infty} \frac{2}{x^2}\,\mathrm{d}x = -\frac{2}{x}\bigg|_1^{\infty} = 2.$$

Hence $g(x) = 1/(x^2+1)$ is integrable on $(0, \infty)$, and $|\frac{\partial}{\partial x} f(n,x)| \leqslant g(x)$ for all x, n. Therefore, the hypotheses of differentiation under the integral sign are satisfied, so

$$f'(x) = \int \frac{\partial}{\partial x} f(n,x)\,\mathrm{d}n = \sum_{n=1}^{\infty} \frac{\partial}{\partial x} f(n,x) = \sum_{n=1}^{\infty} \frac{1-n^2x^2}{(1+n^2x^2)^2},$$

as needed.

Finally, $f'(x)$ is continuous. Indeed, take a point x_0, and let (x_k) be a sequence of points such that $x_k \to x_0$. Define $h_k(n) := \frac{\partial}{\partial x} f(n, x_k)$. Then, since f' is continuous in x_k, we have $h_k(n) \to f(n, x_0)$ for all n. On the other hand, since x, n are symmetric in $\frac{\partial}{\partial x} f(n, x)$, then by an identical proof to the one given above, we have

$$|h_k(n)| \leqslant \frac{1}{1 + n^2 x_k^2}.$$

Since (x_k) is convergent, there is some minimal value y in this sequence. Hence $|h_k(n)| \leqslant 1/(1 + n^2 y)$ for all n. Since $x_0 \neq 0$, and all $x_k \in (0, \infty)$, we can guarantee $y \neq 0$. Now,

$$\int \frac{1}{1 + n^2 y} \, \mathrm{d}n = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 y} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2 y} = \frac{1}{y} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, the h_k are all bounded by a Lebesgue integrable function on $\mathbb{N}$. Therefore, by DCT,

$$f'(x_0) = \int \frac{\partial}{\partial x} f(n, x) \, \mathrm{d}x = \lim_{k \to \infty} \int h_k(n) \, \mathrm{d}n = \lim \int \frac{\partial}{\partial x} f(n, x_k) \, \mathrm{d}n = \lim f'(x_k).$$

So, f' is a continuous function, completing the proof. □

Problem 2.

(a) Let E be a subset of $\mathbb{R}^d$ with the property that $E \cap \{x \in \mathbb{R}^d : |x| \leqslant k\}$ is closed for all $k \in \mathbb{N}$. Prove that E must itself be closed.

(b) Let μ be a Borel measure on $\mathbb{R}^d$ that assigns finite measure to all bounded Borel sets. Prove that for every F_σ set $V \subseteq \mathbb{R}^d$ and $\epsilon > 0$ there exists a closed set $F \subseteq V$ such that $\mu(V/F) < \epsilon$.

Proof. We prove (a) by applying the limit definition for closed sets over metric spaces. Let (x_n) be an E-sequence converging to x_0. Since $|\cdot|$ is continuous, then $|x_n|$ converges to $|x_0|$. Hence, $|x_n|$ is bounded, say by $M \in \mathbb{N}$. The set $E \cap \{x \in \mathbb{R}^d : |x| \leqslant M\}$ is closed, and (x_n) is a convergent sequence in this set, so we conclude that $x_0 \in E \cap \{x \in \mathbb{R}^d : |x| \leqslant M\}$. Therefore, $x_0 \in E$. We have just shown E contains all its limit points, so E is closed.

Let $\epsilon > 0$. Say that $V = \bigcup_{n=1}^{\infty} E_n$ for E_n closed sets. We may suppose the E_n are monotone, since we may write $V = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{m} E_n$, and each $\bigcup_{n=1}^{m} E_n$ is closed, given that finite unions of closed sets are closed. Let $B_r = \{x \in \mathbb{R}^d : |x| \leqslant r\}$. Since $|\cdot|$ is continuous, B_r is closed. Define $S_n = B_n \backslash B_{n-1}(0)$, where $B_{n-1}(0)$ is the open ball at the origin of radius $n-1$. Since $B_{n-1}(0)$ is open, and B_n is closed, then S_n is closed. Moreover, $S_n \subseteq B_n$ is bounded and thus has finite measure. We have $V \cap S_m = \bigcup_{n=1}^{\infty} (E_n \cap S_m)$, with $E_n \cap S_m \subseteq E_{n+1} \cap S_m$ by monotonicity. Hence,

$$\mu(V \cap S_m) = \lim_{n \to \infty} \mu(E_n \cap S_m).$$

Choose k_m so that $\mu((V \cap S_m) \backslash (E_{k_m} \cap S_m)) = \mu(V \cap S_m) - \mu(E_{k_m} \cap S_m) < \epsilon/2^m$, where these inequalities are well-defined by finiteness. Define $A_m = E_{k_m} \cap S_m$ so that $\mu((V \cap S_m) \backslash A_m) < \epsilon/2^m$ for all $m \in \mathbb{N}$. Set $A = \bigcup_{m=1}^{\infty} A_m$. Observe that $V = \bigcup_{m=1}^{\infty} S_m \cap V$. So,

$$V \backslash A = \left(\bigcup_{n=1}^{\infty} V \cap S_n \right) \backslash A \subseteq \bigcup_{n=1}^{\infty} ((V \cap S_n) \backslash A) \subseteq \bigcup_{n=1}^{\infty} ((V \cap S_n) \backslash A_n).$$

Therefore,

$$\mu(V\backslash A) \leqslant \mu\left(\bigcup_{n=1}^{\infty}((V\cap S_n)\backslash A_n)\right) \leqslant \sum_{n=1}^{\infty}\mu((V\cap S_n)\backslash A_n) < \sum_{n=1}^{\infty}\epsilon/2^n = \epsilon.$$

We prove that A is closed. So, note that $B_n \cap S_m = \emptyset$ for all $m > n+1$.

$$A\cap B_n = \bigcup_{m=1}^{\infty}(A_m\cap B_n) = \bigcup_{m=1}^{\infty}(E_{k_m}\cap S_m\cap B_n) = \bigcup_{m=1}^{n+1}(E_{k_m}\cap S_m\cap B_n).$$

Now, $E_{k_m}\cap S_m\cap B_n$ is an intersection of closed sets and hence closed. So, $\bigcup_{m=1}^{n+1}(E_{k_m}\cap S_m\cap B_n)$ is a finite union of closed sets and therefore closed. Thus, $A\cap B_n$ is closed. By (a), we conclude that A is closed. Setting $F = A$ gives the result. □

Problem 3. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}}|f(x)|^{1/n}\,\mathrm{d}x = m(\{x\in\mathbb{R} : f(x)\neq 0\})$$

where we are allowing for the possibility that both sides equal infinity.

Proof. Set $A = \{x\in\mathbb{R} : f(x)\neq 0\}$, set $E = \{x\in\mathbb{R} : |f(x)| < 1\}$, and define $g_n(x) = |f(x)|^{1/n}$. Note that if $f(x)\neq 0$, then $g_n(x)\to 1$. Hence, $g_n \to \mathbb{1}_A$ pointwise. Now, over E, we have $|f(x)|<1$, and so $|f(x)|^{1/n} = g_n(x)$ increases monotonically in n. Therefore, by MCT,

$$\lim\int_E g_n\,\mathrm{d}x = \int_E \mathbb{1}_A\,\mathrm{d}x.$$

On the other hand, on E^c, we have $g_n(x)\leqslant |f(x)|$ for all x. Since $f\in L^1(\mathbb{R})$, then by DCT, we achieve

$$\lim\int_{E^c} g_n\,\mathrm{d}x = \int_{E^c}\mathbb{1}_A\,\mathrm{d}x.$$

Therefore,

$$m(\{x\in\mathbb{R} : f(x)\neq 0\}) = \int \mathbb{1}_A\,\mathrm{d}x = \lim\int_E g_n\,\mathrm{d}x + \lim\int_{E^c} g_n\,\mathrm{d}x = \lim\int g_n\,\mathrm{d}x.$$

Since $g_n = |f|^{1/n}$, the proof is complete. □

Problem 4. Let $f, g \in L^2([0,1])$. Prove that if

$$\int_0^1 f(x)x^n\,\mathrm{d}x = \int_0^1 g(x)x^n\,\mathrm{d}x$$

for all integers $n \geqslant 0$, then $f = g$ almost everywhere.

Proof. We claim that polynomials are dense in $L^2([0,1])$. Let $h \in L^2([0,1])$. Then, since compactly supported continuous functions are dense in $L^2([0,1])$, there is some continuous $k(x)$ such that $||h-k||_2 < \epsilon$. Now, by Weierstrass's Approximation Theorem, there is some polynomial $p(x)$ such that for all $x \in [0,1]$, we have $|k(x) - p(x)| < \epsilon$. Therefore,

$$||k-p||_2 = \left(\int_0^1 |h(x) - p(x)|^2 \, \mathrm{d}x\right)^{1/2} \leqslant \left(\int_0^1 \epsilon^2 \, \mathrm{d}x\right)^{1/2} = \epsilon.$$

So, $||h-p||_2 \leqslant ||h-k||_2 + ||k-p||_2 \leqslant 2\epsilon$. So, polynomials are dense in $L^2([0,1])$.

Since $f - g \in L^1([0,1])$, we choose a sequence $p_n(x)$ of polynomials converging to $f-g$ in $L^2([0,1])$. By anti-linearity of the inner product, $\langle f-g, p_n(x)\rangle = 0$ for all n. Then, by continuity of the inner product and of complex conjugation, we have

$$||f-g||_2^2 = \langle f-g, f-g\rangle = \overline{\lim\langle p_n, f-g\rangle} = \lim\langle f-g, p_n\rangle = \lim 0 = 0.$$

Therefore, $||f-g||_2 = 0$. By Cauchy-Schwarz, we have

$$||f-g||_1 = \int_0^1 |f-g| \cdot 1 \, \mathrm{d}x \leqslant ||f-g||_2 ||1||_2 = ||f-g||_2 = 0.$$

Since $||f-g||_1 = 0$, then $f = g$ almost everywhere. □

Problem 5. Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then

$$f * g(x) := \int f(x-y)g(y) \, \mathrm{d}y$$

defines a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that satisfies the following estimates:

(a) **(Classic)** $||f * g||_1 \leqslant ||f||_1 ||g||_1$,

(b) $||f * g||_2 \leqslant ||g||_1 ||f||_2$.

Hint: For the second estimate, first argue that $|f * g|^2 \leqslant ||g||_1 (|f|^2 * |g|)$.

Proof. We prove (a). So,

$$\int |f * g(x)| \, \mathrm{d}x = \int \left|\int f(x-y)g(y) \, \mathrm{d}y\right| \mathrm{d}x \leqslant \iint |f(x-y)g(y)| \, \mathrm{d}y \, \mathrm{d}x.$$

Applying Tonelli's Theorem,

$$\iint |f(x-y)g(y)| \, \mathrm{d}y \, \mathrm{d}x = \iint |f(x-y)g(y)| \, \mathrm{d}x \, \mathrm{d}y = \int |g(y)| \left(\int |f(x-y)| \, \mathrm{d}x\right) \mathrm{d}y.$$

Then, by translation invariance,

$$\int |g(y)| \left(\int |f(x-y)| \, \mathrm{d}x\right) \mathrm{d}y = \int |g(y)| \int |f(x)| \, \mathrm{d}x \, \mathrm{d}y = \left(\int |g(y)| \, \mathrm{d}y\right) \left(\int |f(x)| \, \mathrm{d}x\right).$$

So, $||f * g||_1 \leqslant ||f||_1 ||g||_1$.

We now prove (b). Since $g \in L^1(\mathbb{R})$, then $|g|^{1/2} \in L^2(\mathbb{R})$. So, fixing $x \in \mathbb{R}$, we apply Cauchy-Schwarz to obtain

$$\int |f(x-y)||g(y)|^{1/2} \cdot |g(y)|^{1/2} \, \mathrm{d}y \leqslant \left(\int |f(x-y)|^2 |g(y)| \, \mathrm{d}y \right)^{1/2} \left(\int |g(y)| \, \mathrm{d}y \right)^{1/2}.$$

The RHS is $|f|^2 * |g|(x)^{1/2} \cdot ||g||_1^{1/2}$. Hence, taking the square on both sides, we obtain

$$|f * g(x)|^2 \leqslant ||g||_1 (|f|^2 * |g|(x)).$$

Furthermore, applying Tonelli's Theorem, we have

$$\begin{aligned} \int |f|^2 * |g|(x) \, \mathrm{d}x &= \iint |f(x-y)|^2 |g(y)| \, \mathrm{d}y \, \mathrm{d}x \\ &= \iint |f(x-y)|^2 \, \mathrm{d}x |g(y)| \, \mathrm{d}y \\ &= \iint |f(x)|^2 \, \mathrm{d}x |g(y)| \, \mathrm{d}y \\ &= ||f||_2^2 \cdot ||g||_1. \end{aligned}$$

Therefore,

$$||f * g||_2 = \left(\int |f * g(x)|^2 \, \mathrm{d}x \right)^{1/2} \leqslant \left(\int ||g||_1 (|f|^2 * |g|(x)) \, \mathrm{d}x \right)^{1/2} = \left(||f||_2^2 \cdot ||g||_1^2 \right)^{1/2}.$$

Hence, $||f * g||_2 \leqslant ||f||_2 ||g||_1$. □

Problem 6.

(a) Prove that if $E \subseteq \mathbb{R}$ with $m(E) > 0$, then

$$\int_E e^{-\pi x^2} \, \mathrm{d}x > 0.$$

(b) **(Classic Technique)** Let $f \in L^\infty(\mathbb{R})$. Prove that

$$\lim_{p \to \infty} \left(\int |f(x)|^p e^{-\pi x^2} \, \mathrm{d}x \right)^{1/p} = ||f||_\infty.$$

Proof. Take $E \subseteq \mathbb{R}$ with $m(E) > 0$. Since $E = \bigcup_{n=1}^\infty ([-n, n] \cap E)$, there is some interval $I = [-n, n]$ such that $E \cap I$ has nonzero measure. Since $e^{-\pi x^2}$ decreases monotonically on $[0, n]$, then $e^{-\pi n^2} = \alpha$ is the minimal value attained by $e^{-\pi x^2}$ on $[0, n]$. Moreover, $e^{-\pi x^2}$ is symmetric about the y-axis, so in fact α is the minimal value attained by $e^{-\pi x^2}$ on all of I. Observe that $\alpha > 0$. Hence, since $e^{-\pi x^2}$ is nonnegative,

$$\int_E e^{-\pi x^2} \, \mathrm{d}x \geqslant \int_{E \cap I} e^{-\pi x^2} \, \mathrm{d}x \geqslant \int_{E \cap I} \alpha \, \mathrm{d}x = m(E \cap I)\alpha > 0,$$

proving part (a).

Moving on to (b), we first have

$$\left(\int |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p} \leqslant \left(\int ||f||_\infty^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p} = ||f||_\infty \left(\int e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p}.$$

Since $\int e^{-\pi x^2}\,\mathrm{d}x = 1$, then $(\int |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x)^{1/p} \leqslant ||f||_\infty$. Set $\alpha = ||f||_\infty - \epsilon$. Let $A = \{x \in \mathbb{R} : f(x) \geqslant \alpha\}$. The measure of A is nonzero, given that $\alpha < ||f||_\infty$. By our argument in (a), there is some subset $B \subseteq A$ such that B has nonzero measure, and $e^{-\pi x^2}$ is bounded below by b on B, with b positive. We then have

$$\alpha(bm(B))^{1/p} = \left(\int_B |\alpha|^p b\,\mathrm{d}x\right)^{1/p} \leqslant \left(\int_B |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p}.$$

By nonnegativity, we obtain

$$\alpha(bm(B))^{1/p} \leqslant \left(\int_B |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p} \leqslant \left(\int |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p} \leqslant ||f||_\infty.$$

Taking p to ∞, we observe that $(bm(B))^{1/p} \to 1$, so that

$$||f||_\infty - \epsilon = \alpha \leqslant \lim_{p\to\infty} \left(\int |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p} \leqslant ||f||_\infty.$$

This holds for all ϵ, so we conclude that

$$||f||_\infty = \lim_{p\to\infty} \left(\int |f(x)|^p e^{-\pi x^2}\,\mathrm{d}x\right)^{1/p},$$

completing the proof. □